Contents

1.1 Complexes of R-Modules

Homological algebra is a tool used in several branches of mathematics: algebraic topology, group theory, commutative ring theory, and algebraic geometry come to mind. It arose in the late 1800s in the following manner. Let f and g be matrices whose product is zero. If $g \cdot v = 0$ for some column vector v, say, of length n, we cannot always write $v = f \cdot u$. This failure is measured by the *defect*

$$d = n - \operatorname{rank}(f) - \operatorname{rank}(g).$$

In modern language, f and g represent linear maps

$$U \xrightarrow{f} V \xrightarrow{g} W$$

with $gf = 0$, and d is the dimension of the *homology module*

$$H = {}^{\ker(g)}\!\big/_{f(U)}.$$

In the first part of this century, Poincaré and other algebraic topologists utilized these concepts in their attempts to describe "n-dimensional holes" in simplicial complexes. Gradually peopled noticed that "vector space" could be replaced by "R-module" for any ring R.

This being said, we fix an associative ring R and begin again in the category **mod**-R of right R-modules. Given an R-module homomorphism $f : A \to B$, one is immediately led to study the kernel $\ker(f)$, cokernel $\operatorname{coker}(f)$, and image $\operatorname{im}(f)$ of f. Given another map $g : B \to C$, we can form the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C. \qquad (*)$$

We say that such a sequence is *exact* (at B) if $\ker(g) = \operatorname{im}(f)$. This implies in particular that the composite $gf : A \to C$ is zero, and finally brings out attention to sequences $(*)$ such that $gf = 0$.

Definition 1.1.1 A *chain complex* $C_\bullet$ of R-modules is a family $\{C_n\}_{n\in\mathbf{Z}}$ of R-modules, together with R-module maps $d = d_n : C_n \to C_{n-1}$ such that each composite $d \circ d : C_n \to C_{n-2}$ is zero. The maps d_n are called the *differentials* of $C_\bullet$. The kernel of d_n is the module of *n-cycles* of $C_\bullet$, denoted $Z_n = Z_n(C_\bullet)$. The image of $d_{n+1} : C_{n+1} \to C_n$ is the module of *n-boundaries* of $C_\bullet$, denoted $B_n = B_n(C_\bullet)$. Because $d \circ d = 0$, we have

$$0 \subseteq B_n \subseteq Z_n \subseteq C_n$$

for all n. The n^{th} *homology module* of $C_\bullet$ is the subquotient $H_n(C_\bullet) = {}^{Z_n}\!/_{B_n}$ of C_n. Because the dot in $C_\bullet$ is annoying, we will often write C for $C_\bullet$.

Exercise 1.1.1 Set $C_n = {}^{\mathbf{Z}}\!/_{8}$ for $n \geq 0$ and $C_n = 0$ for $n < 0$; for $n > 0$ let d_n send $x \pmod 8$ to $4x \pmod 8$. Show that $C_\bullet$ is a chain complex of ${}^{\mathbf{Z}}\!/_{8}$-modules and compute its homology modules.

We must show that $d_n \circ d_{n+1} = 0$. Let $x \in C_{n+1} = {}^{\mathbf{Z}}\!/_{8}$. Then $(d_n \circ d_{n+1})(x) = d_n(4x \pmod 8) = 16x \pmod 8 \equiv 0$, as desired.

To compute $H_n(C_\bullet)$ for all $n > 0$, see that $\ker(d_n) = \{0, 2, 4, 6\} \cong {}^{\mathbf{Z}}\!/_{4}$ and $\operatorname{im}(d_{n+1}) = \{0, 4\} \cong {}^{\mathbf{Z}}\!/_{2}$, so $H_n(C_\bullet) = {}^{{}^{\mathbf{Z}}\!/_{4}}\!\big/_{{}^{\mathbf{Z}}\!/_{2}} = {}^{\mathbf{Z}}\!/_{2}$.

For $n = 0$, $H_0(C_\bullet) = {}^{\ker(d_0)}\!\big/_{\operatorname{im}(d_1)} = {}^{{}^{\mathbf{Z}}\!/_{8}}\!\big/_{{}^{\mathbf{Z}}\!/_{2}} = {}^{\mathbf{Z}}\!/_{4}$.

For $n < 0$, $H_n(C_\bullet) = {}^{0}\!/_{0} = 0$.

There is a category **Ch**(**mod**-R) of chain complexes of (right) R-modules. The objects are, of course, chain complexes. A *morphism* $u : C \to D$ is a chain complex map, that is, a family of R-module homomorphisms $u_n : C_n \to D_n$ commuting with d in the sense that $u_{n-1}d_n = d_n u_n$. That is, such that the following diagram commutes

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d} & C_{n+1} & \xrightarrow{d} & C_n & \xrightarrow{d} & C_{n-1} & \xrightarrow{d} & \cdots \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u} & & \\ \cdots & \xrightarrow{d} & D_{n+1} & \xrightarrow{d} & D_n & \xrightarrow{d} & D_{n-1} & \xrightarrow{d} & \cdots . \end{array}$$

Exercise 1.1.2 Show that a morphism $u : C \to D$ of chain complexes sends boundaries to boundaries and cycles to cycles, hence maps $H_n(C_\bullet) \to H_n(D_\bullet)$. Prove that each H_n is a functor from **Ch**(**mod**-R) to **mod**-R.

Let $x \in Z_n(C_\bullet)$. Then $x \in \ker(d_n)$, so $d_n(x) = 0$. As u_n are R-module homomorphisms, $u_{n-1}d_n(x) = 0$. So $d_n u_n(x) = 0$, and thus $u_n(x) \in \ker(d_n) = Z_n(D_\bullet)$.

Let $y \in B_n(C_\bullet)$. Then $y \in \operatorname{im}(d_{n+1})$, so $y = d_{n+1}(x)$ for some $x \in C_{n+1}$. We need to show $u_n(y) \in \operatorname{im}(d_{n+1}) = B_n(D_\bullet)$. Since $u_n(y) = u_n d_{n+1}(x) = d_{n+1}u_{n+1}(x)$, $u_n(y) \in B_n(D_\bullet)$, as desired.

Now, we show that each H_n is a functor **Ch**(**mod**-R)→**mod**-R, so fix an arbitrary n. The definition of a functor is that we must show H_n sends objects in **Ch**(**mod**-R) to objects in **mod**-R and assigns $u : C_\bullet \to D_\bullet$ in **Ch**(**mod**-R) to R-module maps $H_n(u)$ such that $H_n(\mathrm{id}_X) = \mathrm{id}_{H_n(X)}$ and $H_n(u \circ v) = H_n(u) \circ H_n(v)$. The first is easy: $H_n(C_\bullet)$ is an R-module, because it's a subquotient ${}^{Z_n}\!/_{B_n} \subseteq C_n$. The second is also easy: see that $H_n(u) = u_n : C_n \to D_n$, so that $H_n(\mathrm{id}_{C_\bullet}) = \mathrm{id} : C_n \to C_n$ and $H_n(u \circ v) = u_n \circ v_n = H_n(u) \circ H_n(v)$.

Exercise 1.1.3 (Split exact sequences of vector spaces) Choose vector spaces $\{B_n, H_n\}_{n\in\mathbf{Z}}$ over a field, and set $C_n = B_n \oplus H_n \oplus B_{n-1}$. Show that the projection-inclusions $C_n \to B_{n-1} \subseteq C_{n-1}$ make $\{C_n\}$ into a chain complex, and that every chain complex of vector spaces is isomorphic to a complex of this form.

With $d_n : C_n \to B_{n-1}$ the projection-inclusion map, we must show that $d_n \circ d_{n+1} = 0$. Let $(x, y, z) \in C_{n+1} = B_n \oplus H_n \oplus B_{n-1}$. Then $(d_n \circ d_{n+1})(x, y, z) = d_n(z, 0, 0) = (0, 0, 0)$, so $\{C_n, d_n\}$ is a chain complex, as desired.

We now need to show that every chain complex of vector spaces is isomorphic to a complex of this form; that is, we must show given any chain complex of vector spaces

$$\cdots \xrightarrow{\partial_{n+2}} V_{n+1} \xrightarrow{\partial_{n+1}} V_n \xrightarrow{\partial_n} V_{n-1} \xrightarrow{\partial_{n-1}} \cdots ,$$

there exists maps $u_n : V_n \to B_n \oplus H_n \oplus B_{n-1}$ that are isomorphisms of vector spaces and that commute with ds and ∂s. So we need to search for a way to decompose a vector space as a direct sum in this way. I am stuck and going off the wild whim chance that the choice of letters B and H is not arbitrary. Let's hope that by B and H we mean boundaries and cycles mod boundaries. Then $B_n = \mathrm{im}(\partial_{n+1})$, $H_n = {}^{\ker(\partial_n)}\!\diagup_{\mathrm{im}(\partial_{n+1})}$, and $B_{n-1} = \mathrm{im}(\partial_n)$. We now seek an isomorphism

$$u_n : V_n \to \mathrm{im}(\partial_{n+1}) \oplus {}^{\ker(\partial_n)}\!\diagup_{\mathrm{im}(\partial_{n+1})} \oplus \mathrm{im}(\partial_n).$$

Since we know that $\{V, \partial\}$ is a chain complex, we can have a bijection, since an element of V_n is either an n-boundary, an n-cycle mod n-boundary, or an $(n-1)$-boundary. Promising!

As with most things vector spaced, we appeal to a suitable basis. Let's choose a basis $\mathcal{B}$ for V_n so that we may write $v \in V_n$ as $t \oplus t' \oplus t''$, with $t \in B_n$, $t' \in H_n$, $t'' \in B_{n-1}$ and each factor a linear combination of basis vectors. (This should be okay to do since we know u_n can be a bijection.) Now explicitly define u_n by $u_n(v) = t \oplus t' + \mathrm{im}(\partial_{n+1}) \oplus \partial_n(t'')$, because that's the only way I can think of to have u_n hit the correct image.

To see that u_n is an isomorphism, see that if $v = t \oplus t' \oplus t''$ and $w = s \oplus s' \oplus s''$, then

$$\begin{aligned}
u_n(v + w) &= t + s \oplus (t' + s') + \mathrm{im}(\partial_{n+1}) \oplus \partial_n(t'' + s'') \\
&= t + s \oplus t' + \mathrm{im}(\partial_{n+1}) + s' + \mathrm{im}(\partial_{n+1}) \oplus \partial_n(t'') + \partial_n(s'') \\
&= \big(t \oplus t' + \mathrm{im}(\partial_{n+1}) \oplus \partial_n(t'')\big) + \big(s \oplus s' + \mathrm{im}(\partial_{n+1}) \oplus \partial_n(s'')\big) \\
&= u_n(v) + u_n(w)
\end{aligned}$$

and if c is in my field,

$$\begin{aligned}
u_n(cv) &= ct \oplus ct' + \mathrm{im}(\partial_{n+1}) \oplus \partial_n(ct'') \\
&= ct \oplus c(t' + \mathrm{im}(\partial_{n+1})) \oplus c\partial_n(t'') \\
&= c\big(t \oplus t' + \mathrm{im}(\partial_{n+1}) \oplus \partial_n(t'')\big) \\
&= cu_n(v).
\end{aligned}$$

Finally, we need to show that given $v \in V_n$, $u_{n-1}\partial_n(v) = d_n u_n(v)$. See that $u_{n-1}\partial_n(v)$ must be written uniquely as some $\alpha \oplus \beta \oplus \gamma$, but since $\partial_n(v) \in \operatorname{im}(\partial_n)$ obviously, we can thus write $u_{n-1}\partial_n(v) = \partial_n(v) \oplus 0 \oplus 0$. On the other hand, $d_n u_n(v) = d_n(t \oplus t' + \operatorname{im}(d_{n+1}) \oplus d_n(t'')) = d_n(t'') \oplus 0 \oplus 0$. It therefore suffices to show that $\partial_n(v) = d_n(t'')$.

WISHY WASHY Since we can think of $\partial_n(v)$ as $d_n(t, t', t'') = (d_n t'', 0, 0)$, we are done.

Exercise 1.1.4 Show that $\{\operatorname{Hom}_R(A, C_n)\}$ forms a chain complex of abelian groups for every R-module A and every R-module chain complex C. Taking $A = Z_n$, show that if $H_n(\operatorname{Hom}_R(Z_n, C)) = 0$, then $H_n(C) = 0$. Is the converse true?

Let $C_\bullet$ have differentials $\{\partial_n\}$. The maps $d_n : \{\operatorname{Hom}_R(A, C_n)\} \to \{\operatorname{Hom}_R(A, C_{n-1})\}$ are

$$d_n(A \xrightarrow{f} C_n) = A \xrightarrow{f} C_n \xrightarrow{\partial_n} C_{n-1},$$

really the only thing that they could be. Now see that $d_n \circ d_{n+1} = 0$. Indeed,

$$(d_n \circ d_{n+1})(f) = d_n(\partial_{n+1} f) = \partial_n \partial_{n+1} f = 0(f) = 0,$$

since $\{\partial_n\}$ are differentials.

Now let $A = Z_n$ and suppose $H_n(\operatorname{Hom}_R(Z_n, C)) = 0$. Thus

$$\ker(d_n : \{\operatorname{Hom}_R(Z_n, C_n)\} \to \{\operatorname{Hom}_R(Z_{n-1}, C_{n-1})\}) = \operatorname{im}(d_{n+1} : \{\operatorname{Hom}_R(Z_{n+1}, C_{n+1})\} \to \{\operatorname{Hom}_R(Z_n, C_n)\})$$

so equivalently

$$\{f \in \operatorname{Hom}_R(Z_n, C_n) \mid d_n(f) = 0\} = \{g \in \operatorname{Hom}_R(Z_n, C_n) \mid g = d_{n+1}(\widetilde{g}) \text{ for some } \widetilde{g} \in \operatorname{Hom}_R(Z_{n+1}, C_{n+1})\}$$

so equivalently

$$\{f \in \operatorname{Hom}_R(Z_n, C_n) \mid \partial_n f = 0\} = \{g \in \operatorname{Hom}_R(Z_n, C_n) \mid g = \partial_{n+1}\widetilde{g} \text{ for some } \widetilde{g} \in \operatorname{Hom}_R(Z_{n+1}, C_{n+1})\}.$$

We need to show $Z_n = B_n$; since $B_n \subseteq Z_n$ always, it is enough to show $Z_n \subseteq B_n$. Let $x \in Z_n$. Then $\partial_n(x) = 0$, so $(\partial_n \circ i)(x) = 0$, where $i : Z_n \hookrightarrow C_n$. So $i \in \ker(d_n) = \operatorname{im}(d_{n+1})$, which means $i = \partial_{n+1}\widetilde{g}$ for some $\widetilde{g} : Z_{n+1} \to C_{n+1}$. Therefore, $x = i(x) = \partial_{n+1}\widetilde{g}(x)$, and therefore $x \in \operatorname{im}(\partial_{n+1}) = B_n$, and the claim is proven.

The converse is also true. If $H_n(C) = 0$, then $Z_n = B_n$. We need to show that $\ker(d_n) = \{f \mid d_n(f) = 0\} = \{g \mid d_{n+1}(\widetilde{g}) = g \text{ for some } \widetilde{g}\} = \operatorname{im}(d_{n+1})$. Just as $B_n \subseteq Z_n$ always, $\operatorname{im}(d_{n+1}) \subseteq \ker(d_n)$, so we show $\ker(d_n) \subseteq \operatorname{im}(d_{n+1})$. Let $f \in \ker(d_n)$, so $f : Z_n \to C_n$ such that $d_n(f) = 0$. Thus $\partial_n f(x) = 0$ for all x, so $f(x) \in Z_n$ which is B_n by hypothesis. Therefore, there exists $\widetilde{g}$ such that $\partial_{n+1}\widetilde{g}(f(x)) = \partial_{n+1}\widetilde{g}f(x) = f(x)$, so $d_{n+1}(\widetilde{g}f) = f$, and therefore $f \in \text{im}(d_{n+1})$. Thus, $\ker(d_n) = \text{im}(d_{n+1})$ and therefore $H_n(\text{Hom}_R(Z_n, C)) = 0$, as claimed.

Definition 1.1.2 A morphism $C_\bullet \to D_\bullet$ of chain complexes is called a *quasi-isomorphism* (Bourbaki uses *homologism*) if the maps $H_n(C_\bullet) \to H_n(D_\bullet)$ are all isomorphisms.

Exercise 1.1.5 Show that the following are equivalent for every $C_\bullet$:

1. $C_\bullet$ is *exact*, that is, exact at every C_n.
2. $C_\bullet$ is *acyclic*, that is, $H_n(C_\bullet) = 0$ for all n.
3. The map $0 \to C_\bullet$ is a quasi-isomorphism, where "0" is the complex of zero modules and zero maps.

First, 1. implies 2.: If $C_\bullet$ is exact, then at C_n, $Z_n = \ker(d_n) = \text{im}(d_{n+1}) = B_n$ for all n. Then, $H_n(C_\bullet) = {}^{Z_n}\!/\!_{B_n} = {}^{Z_n}\!/\!_{Z_n} = 0$, so $C_\bullet$ is acyclic.

Next, 2. implies 3.: We need to show $0 \to C_\bullet$ is a quasi-isomorphism; we need to show $H_n(0) \xrightarrow{\sim} H_n(C_\bullet)$ for every n. Since for all n, $H_n(0) = 0$ obviously and $H_n(C_\bullet) = 0$ for all n by hypothesis, the only map $0 \to 0$ is an isomorphism, and we are done.

Finally, 3. implies 1.: Given a quasi-isomorphism $0 \to C_\bullet$, for each n, $0 = H_n(C_\bullet) = {}^{Z_n}\!/\!_{B_n}$. Since $B_n \subseteq Z_n$, $Z_n = B_n$, and thus $C_\bullet$ is exact.

The following variant notation is obtained by reindexing with superscripts: $C^n = C_{-n}$. A *cochain complex* $C^\bullet$ of R-modules is a family $\{C^n\}$ of R-modules, together with maps $d^n : C^n \to C^{n+1}$ such that $d \circ d = 0$. $Z^n(C^\bullet) = \ker(d^n)$ is the module of *n-cocycles*, $B^n(C^\bullet) = \text{im}(d^{n-1}) \subseteq C^n$ is the module of *n-coboundaries*, and the subquotient $H^n(C^\bullet) = {}^{Z^n}\!/\!_{B^n}$ of C^n is the n^{th} *cohomology module* of $C^\bullet$. Morphisms and quasi-isomorphisms of cochain complexes are defined exactly as for chain complexes.

A chain complex $C_\bullet$ is called *bounded* if almost all the C_n are zero; if $C_n = 0$ unless $a \le n \le b$, we say that the complex has *amplitude* in $[a, b]$. A complex $C_\bullet$ is *bounded above* (resp. *bounded below*) if there is a bound b (resp. a) such that $C_n = 0$ for all $n > b$ (resp. $n < a$). The bounded (resp. bounded above, resp. bounded below) chain complexes form full subcategories of $\mathbf{Ch}$=$\mathbf{Ch}(R\text{-}\mathbf{mod})$ that are denoted $\mathbf{Ch}_b$, $\mathbf{Ch}_-$, and $\mathbf{Ch}_+$, respectively. The subcategory $\mathbf{Ch}_{\ge 0}$ of non-negative complexes $C_\bullet$ ($C_n = 0$ for all $n < 0$) will be important in Chapter 8.

Similarly, a cochain complex $C^\bullet$ is called *bounded above* if the chain complex $C_\bullet$ ($C_n = C^{-n}$) is bounded below, that is, if $C^n = 0$ for all large n; $C^\bullet$ is *bounded below* if $C_\bullet$ is bounded above, and *bounded* if $C_\bullet$ is bounded. The categories of bounded (resp. bounded above, resp. bounded below, resp. non-negative) cochain complexes are denoted $\mathbf{Ch}^b$, $\mathbf{Ch}^-$, $\mathbf{Ch}^+$, and $\mathbf{Ch}^{\ge 0}$, respectively.

Exercise 1.1.6 (Homology of a graph) Let Γ be a finite graph with V vertices $(v_1, ..., v_V)$ and E edges $(e_1, ..., e_E)$. If we orient the edges, we can form the *incidence matrix* of the graph. This is a $V \times E$ matrix whose (ij) entry is $+1$ if the edge e_j starts at v_i, -1 if e_j ends at v_i, and 0 otherwise. Let C_0 be the free R-module on the vertices, C_1 the free R-module on the edges, $C_n = 0$ if $n \ne 0, 1$, and $d : C_1 \to C_0$ be the incidence matrix. If Γ is connected (i.e., we can get from v_0 to every other vertex by tracing a path with edges), show that $H_0(C)$ and $H_1(C)$ are free R-modules of dimensions

1 and $E-V+1$ respectively. (The number $E-V+1$ is the number of *circuits* of the graph.) *Hint*: Choose basis $\{v_1, v_1 - v_1, ..., v_V - v_1\}$ for C_0, and use a path from v_1 to v_i to find an element of C_1 mapping to $v_i - v_1$.

We need to compute the image and the kernel of d. By construction, $C_0 = R^V$ and $C_1 = R^E$. For $\mathrm{im}(d)$, as per the hint, denote a basis for R^V by fixing a vertex v_0 and taking the set $\{v_0, v_1 - v_0, ..., v_V - v_0\}$. We're going to show that given any $v_i - v_0 \in C_0$, there exists an element in C_1 that maps to it, leaving $H_0(C) = {}^{\ker(C_0 \to 0)}\!\big/\!_{\operatorname{im} d} = {}^{C_0}\!\big/\!_{\langle v_i - v_0\rangle} = \langle v_0 \rangle = R^1$. To do this, fix $v_i - v_0$. Since Γ is path connected, there exists a directed path connecting v_i and v_0, which we may write as $f_\ell + \cdots + f_k$, where $f_j = \pm e_j$, depending on the orientation of each e_j so that the endpoints line up and the path is nicely defined. We claim $d(f_\ell + \cdots + f_k) = v_i - v_0$. See that

$$\begin{aligned}
d(f_\ell + \cdots + f_k) &= d(f_\ell) + \cdots + d(f_k) \\
&= d(\pm e_\ell) + \cdots + d(\pm e_k) \\
&= \pm d(e_\ell) \pm \cdots \pm d(e_k) \\
&= (v_{j_1} - v_0) + (v_{j_2} - v_{j_1}) + \cdots + (v_i - v_{j_m}),
\end{aligned}$$

which telescopes to $-v_0 + v_i$, as desired.

Now, $\ker d$, is easy. Note that as $0 \to C_1$, $H_1(C) = {}^{\ker d}\!\big/\!_{0} = \ker d$. Since $\ker d \leq C_1 = R^E$, $\ker d$ is free, since R^E is free as a group. Thus, by rank-nullity,

$$E = \operatorname{rank}(\ker d) + \operatorname{rank}(\operatorname{im} d)$$

$$\begin{aligned}
&= \operatorname{rank}(H_1(C)) + \operatorname{rank}(C_0) - \operatorname{rank}\left({}^{C_0}\!\big/\!_{\operatorname{im} d}\right) \\
&= \operatorname{rank}(H_1(C)) + \operatorname{rank}(C_0) - \operatorname{rank}(H_0(C)) \\
&= \operatorname{rank}(H_1(C)) + V - 1.
\end{aligned}$$

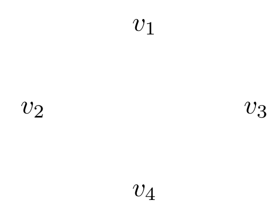
So $E = \operatorname{rank}(H_1(C)) + V - 1$, and thus $H_1(C) = R^{E-V+1}$, as desired.

Application 1.1.3 (Simplicial homology) Here is a topological application we shall discuss more in Chapter 8. Let K be a geometric simplicial complex, such as a triangulated polyhedron, and let K_k $(0 \leq k \leq n)$

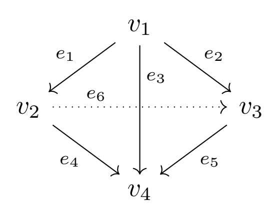
denote the set of k-dimensional simplices of K. Each k-simplex has $k+1$ faces, which are ordered if the set K_0 of vertices is ordered (do so!), so we obtain $k+1$ set maps $\partial_i : K_k \to K_{k-1}$ $(0 \le i \le k)$. The *simplicial chain complex* of K with coefficients in R is the chain complex $C_\bullet$, formed as follows. Let C_k be the free R-module on the set K_k; set $C_k = 0$ unless $0 \le k \le n$. The set maps ∂_i yield $k+1$ module maps $C_k \to C_{k-1}$, which we also call ∂_i; their alternating sum $d = \sum(-1)^i \partial_i$ is the map $C_k \to C_{k-1}$ in the chain complex $C_\bullet$. To see that $C_\bullet$ is a chain complex, we need to prove the algebraic assertion that $d \circ d = 0$. This translates into the geometric fact that each $(k-2)$-dimensional simplex contained in a fixed k-simplex σ of K lies on exactly two faces of σ. The homology of the chain complex $C_\bullet$ is called the *simplicial homology* of K with coefficients in R. This simplicial approach to homology was used in the first part of this century, before the advent of singular homology.

Exercise 1.1.7 (Tetrahedron) The tetrahedron T is a surface with 4 vertices, 6 edges, and 4 2-dimensional faces. Thus its homology is the homology of a chain complex $0 \to R^4 \to R^6 \to R^4 \to 0$. Write down the matrices in this complex and verify computationally that $H_2(T) \cong H_0(T) \cong R$ and $H_1(T) = 0$.

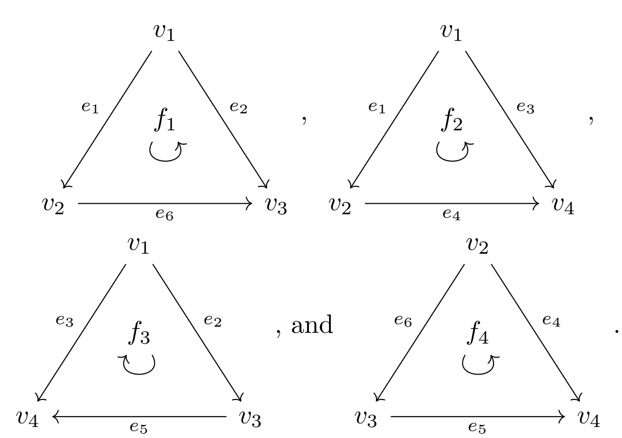
Order our vertices v_1, v_2, v_3, v_4.



This forces an orientation on edges; direct the edge toward the higher indexed vertex.



And it forces an orientation on faces; the face's orientation agrees with as many edges as it can. The picture below shows each of the four faces.



Thus, generalizing (ij) is 1 if e_j starts at v_i to $(ij) = 1$ if f_j flows with e_i (and -1 if the orientations are against one another), we can denote the matrices as follows:

$$d_1 : R^6 \to R^4 \text{ is } \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & -1 & 0 \end{bmatrix}, \text{ and}$$

$$d_2 : R^4 \to R^6 \text{ is } \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Now, we can verify computationally that $H_2(T) \cong H_0(T) \cong R$ and $H_1(T) = 0$. See that

$$d_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

so by rank-nullity, $\dim \operatorname{im} d_1 + \dim \ker d_1 = 6$, and so $\dim \operatorname{im} d_1 = \dim \ker d_1 = 3$. For d_2,

$$d_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so $\dim \operatorname{im} d_2 + \dim \ker d_2 = 4$, and so $\dim \operatorname{im} d_2 = 3$ and $\dim \ker d_2 = 1$. Therefore,

$$\begin{aligned}
H_0(T) &= {}^{\ker(R^4 \to 0)}\!/\!_{\operatorname{im}(d_1)} = {}^{R^4}\!/\!_{R^3} = R, \\
H_1(T) &= {}^{\ker d_1}\!/\!_{\operatorname{im} d_2} = {}^{R^3}\!/\!_{R^3} = 0, \text{ and} \\
H_2(T) &= {}^{\ker d_2}\!/\!_{\operatorname{im}(0 \to R^4)} = {}^{R}\!/\!_{0} = R.
\end{aligned}$$

Application 1.1.4 (Singular homology) Let X be a topological space, and let $S_k = S_k(X)$ be the free R-module on the set of continuous maps from the standard k-simplex Δ_k to X. Restriction to the i^{th} face of Δ_k $(0 \le i \le k)$ transforms a map $\Delta_k \to X$ into a map $\Delta_{k-1} \to X$, and induces an R-module homomorphism ∂_i from S_k to S_{k-1}. The alternating sums $d = \sum (-1)^i \partial_i$ (from S_k to S_{k-1}) assemble to form a chain complex

$$\cdots \xrightarrow{d} S_2 \xrightarrow{d} S_1 \xrightarrow{d} S_0 \to 0,$$

called the *singular chain complex* of X. The n^{th} homology module of $S_\bullet(X)$ is called the *n^{th} singular homology* of X (with coefficients in R) and is written $H_n(X; R)$. If X is a geometric simplicial complex, then the obvious inclusion $C_\bullet(X) \to S_\bullet(X)$ is a quasi-isomorphism, so the simplicial and singular homology modules of X are isomorphic. The interested reader may find details in any standard book on algebraic topology.

1.2 Operations on Chain Complexes

The main point of this section will be that chain complexes form an abelian category. First we need to recall what an abelian category is. A reference for these definitions is [MacCW].

A category $\mathcal{A}$ is called an **Ab**-*category* if every hom-set $\operatorname{Hom}_{\mathcal{A}}(A, B)$ in $\mathcal{A}$ is given the structure of an abelian group in such a way that composition distributes over addition. In particular, given a diagram in $\mathcal{A}$ of the form

$$A \xrightarrow{f} B \overset{g'}{\underset{g}{\rightrightarrows}} C \xrightarrow{h} D$$

we have $h(g + g')f = hgf + hg'f$ in $\operatorname{Hom}(A, D)$. The category **Ch** is an **Ab**-category because we can add chain maps degreewise; if $\{f_n\}$ and $\{g_n\}$ are chain maps from $C_\bullet$ to $D_\bullet$, their sum is the family of maps $\{f_n + g_n\}$.

An *additive functor* $F : \mathcal{B} \to \mathcal{A}$ between **Ab**-categories $\mathcal{B}$ and $\mathcal{A}$ is a functor such that each $\operatorname{Hom}_{\mathcal{B}}(B', B) \to \operatorname{Hom}_{\mathcal{A}}(FB', FB)$ is a group homomorphism.

An *additive category* is an **Ab**-category $\mathcal{A}$ with a zero object (i.e., an object that is initial and terminal) and a product $A \times B$ for every pair A, B of objects in $\mathcal{A}$. This structure is enough to make finite products the same as finite coproducts. The zero object in **Ch** is the complex "0" of zero modules and maps. Given a family $\{A_\alpha\}$ of complexes of R-modules, the product $\prod A_\alpha$ and coproduct (direct sum) $\oplus A_\alpha$ exist in **Ch** and are defined degreewise: the differentials are the maps

$$\prod d_\alpha : \prod_\alpha A_{\alpha,n} \to \prod_\alpha A_{\alpha,n-1} \qquad \text{and} \qquad \bigoplus d_\alpha : \bigoplus_\alpha A_{\alpha,n} \to \bigoplus_\alpha A_{\alpha,n-1},$$

respectively. These suffice to make **Ch** into an additive category.

Exercise 1.2.1 Show that direct sum and direct product commute with homology, that is, that $\oplus H_n(A_\alpha) \cong H_n(\oplus A_\alpha)$ and $\prod H_n(A_\alpha) \cong H_n(\prod A_\alpha)$ for all n.

Write $([a_\alpha]_1)_\alpha$ to mean $(..., a_\alpha + B_n(A_\alpha), ...)_\alpha$ and write $[(a_\alpha)_\alpha]_2$ to mean $(..., a_\alpha, ...)_\alpha + B_n(\oplus A_\alpha)$. Define a map $\varphi : \oplus H_n(A_\alpha) \to H_n(\oplus A_\alpha)$ by $\varphi\Big(([a_\alpha]_1)_\alpha\Big) = [(a_\alpha)_\alpha]_2$. We should check that φ is well-defined.

Then, to see that φ is an isomorphism, see that

$$\begin{aligned}
\varphi\Big(([a_\alpha]_1)_\alpha + ([b_\alpha]_1)_\alpha\Big) &= \varphi\Big(([a_\alpha]_1 + [b_\alpha]_1)_\alpha\Big) \\
&= \varphi\Big(([a_\alpha + b_\alpha]_1)_\alpha\Big) \\
&= [(a_\alpha + b_\alpha)_\alpha]_2 \\
&= [(a_\alpha)_\alpha + (b_\alpha)_\alpha]_2 \\
&= [(a_\alpha)_\alpha]_2 + [(b_\alpha)_\alpha]_2 \\
&= \varphi\Big(([a_\alpha]_1)_\alpha\Big) + \varphi\Big(([b_\alpha]_1)_\alpha\Big)
\end{aligned}$$

and

$$\begin{aligned}
\varphi\Big(r([a_\alpha]_1)_\alpha\Big) &= \varphi\Big((r[a_\alpha]_1)_\alpha\Big) \\
&= \varphi\Big(([ra_\alpha]_1)_\alpha\Big) \\
&= [(ra_\alpha)_\alpha]_2 \\
&= [r(a_\alpha)_\alpha]_2 \\
&= r[(a_\alpha)_\alpha]_2 \\
&= r\varphi\Big(([a_\alpha]_1)_\alpha\Big)
\end{aligned}$$

and φ has inverse $[(a_\alpha)_\alpha]_2 \mapsto ([a_\alpha]_1)_\alpha$.

Everything works the same for products.

...?

Here are some important constructions on chain complexes. A chain complex B is called a *subcomplex* of C if each B_n is a submodule of C_n and the differential on B is the restriction of the differential on C, that is, when the inclusions $i_n : B_n \subseteq C_n$ constitute a chain map $B \to C$. In this case we can assemble the quotient modules ${C_n}/{B_n}$ into a chain complex

$$\cdots \to {C_{n+1}}/{B_{n+1}} \xrightarrow{d} {C_n}/{B_n} \xrightarrow{d} {C_{n-1}}/{B_{n-1}} \xrightarrow{d} \cdots$$

denoted ${C}/{B}$ and called the *quotient complex*. If $f : B \to C$ is a chain map, the kernels $\{\ker(f_n)\}$ assemble to form a subcomplex of B denoted $\ker(f)$, and the cokernels $\{\operatorname{coker}(f_n)\}$ assemble to form a quotient complex of C denoted $\operatorname{coker}(f)$.

Definition 1.2.1 In any additive category $\mathcal{A}$, a *kernel* of a morphism $f : B \to C$ is defined to be a map $i : A \to B$ such that $fi = 0$ and that is universal with respect to this property[1]. Dually, a *cokernel* of f is a map $e : C \to D$, which is universal with respect to having $ef = 0$. In $\mathcal{A}$, a map $i : A \to B$ is *monic* if $ig = 0$ implies $g = 0$ for every map $g : A' \to A$, and a map $e : C \to D$ is an *epi* if $he = 0$ implies $h = 0$ for every map $h : D \to D'$. (The definition of monic and epi in a non-additive category is slightly different; see A.1 in the Appendix.) It is easy to see that every kernel is monic and that every cokernel is an epi (exercise!).

Exercise 1.2.2 In the additive category $\mathcal{A} = R$-**mod**, show that:

1. The notions of kernels, monics, and monomorphisms are the same.
2. The notions of cokernels, epis, and epimorphisms are also the same.

(Recall that) a monomorphism is a map $i : A \to B$ such that for all $h_1, h_2 : A' \to A$, $ih_1 = ih_2$ implies $h_1 = h_2$. An epimorphism is a map $e : C \to D$ such that for all $j_1, j_2 : D \to D'$, $j_1 e = j_2 e$ implies $j_1 = j_2$. In nice cases, monomorphism just means injective and epimorphism is surjective, so let's show that first.

We need to show a map is a monomorphism if and only if it is injective. Assume $i : A \to B$ is a monomorphism. Then let $h_1 : \ker i \hookrightarrow A$ be the inclusion of the kernel and let $h_2 : \ker i \to A$ be the zero map. Then for all $x \in \ker i$, $ih_1(x) = i(x) = 0 = i(0) = ih_2(x)$, so since i is a monomorphism, $h_1 = h_2$, and thus $\ker i = \operatorname{im}(h_1) = \operatorname{im}(h_2) = 0$, so i is injective.

Now assume $i : A \to B$ is injective. Then there exists a left inverse $\ell : B \to A$ such that $\ell i = \operatorname{id}_A$. Let $ih_1 = ih_2$; we show $h_1 = h_2$. See that $ih_1 = ih_2$ implies $\ell ih_1 = \ell ih_2$, so $\operatorname{id}_A h_1 = \operatorname{id}_A h_2$, so $h_1 = h_2$, as desired.

1. We're going to do this incredibly inefficiently. That is to say, rather than a cycle, we'll do this:

$$\text{monic} \rightleftarrows \text{monomorphism} \rightleftarrows \text{kernel}$$

First, we show $i : A \to B$ monomorphism implies i is monic. Let g be any $A' \to A$ and suppose $ig = 0$. Since also $i0 = 0$ and i is a monomorphism, $ig = i0$ implies $g = 0$, so i is monic.

Now we show $i : A \to B$ monic implies i is a monomorphism. Let $ih_1 = ih_2$; we need to show $h_1 = h_2$. See that $0 = ih_2 - ih_1 = i(h_2 - h_1)$, since R-**mod** is an additive category, hence an **AB**-category. Since i is monic, $0 = h_2 - h_1$, so $h_1 = h_2$, as desired.

[1] So this means that for all maps $n : N \to B$ such that $fn = 0$ there exists a unique map $u : N \to A$ such that $iu = n$.

Now we show $i : A \to B$ the kernel of some $f : B \to C$ implies i is a monomorphism. Let $ih_1 = ih_2 : A' \to B$. Then $fih_1 = fih_2 = 0 : A' \to C$, so by the universal property of the kernel, there exists a unique map $u : A' \to A$ such that $iu = ih_1 = ih_2$, so $(u =)h_1 = h_2$ by uniqueness, as desired.

Finally, we show $i : A \to B$ monomorphism implies i is the kernel of some function $f : B \to C$. I am so stuck.

According to Lance's hint, this problem is equivalent to showing that a map $i : A \to B$ of R-modules is injective if and only if $\ker i = 0$ if and only if $0 \to A \to B$ is exact. I see the parallels (obviously monomorphism if and only if injective) but not necessarily the other explicit connections.

But we can proceed. $0 \to A \xrightarrow{i} B$ exact (at A) if and only if $\ker i = \operatorname{im}(0 \to A) = 0$. Now injective if and only if $\ker i = 0$ is a classic undergrad proof: If $\ker i = 0$, see that $i(x) = i(y)$ if and only if $i(x - y) = 0$, so $x - y \in \ker i = 0$, so $x = y$, and i is injective. If i is injective, see that $0 \subseteq \ker i$ always, and for the other direction, if $x \in \ker i$ means $i(x) = 0 = i(0)$, so by injectivity, $x = 0$, and $\ker i = 0$ as desired.

2. Once part 1 is fleshed out better, part 2 is going to be exactly the same, but with surjective/cokernels/epis/arrows reversed.

Exercise 1.2.3 Suppose that $\mathcal{A} = \mathbf{Ch}$ and f is a chain map. Show that the complex $\ker(f)$ is a kernel of f and that $\operatorname{coker}(f)$ is a cokernel of f.

Let $f : B_\bullet \to C_\bullet$ be the chain map. Then $\ker(f)$ is by definition the subcomplex of B

$$\cdots \to \ker(f_{n+1}) \xrightarrow{d|_{\ker(f_{n+1})}} \ker(f_n) \xrightarrow{d|_{\ker(f_n)}} \ker(f_{n-1}) \to \cdots .$$

We need to show that $\ker(f)$ is a kernel of f, but that doesn't quite make sense, because the kernel is a map $i : A_\bullet \to B_\bullet$ and $\ker(f)$ is a subcomplex. There is a natural map $\ker(f) \overset{i}{\hookrightarrow} B_\bullet \xrightarrow{f} C_\bullet$, so maybe we mean this? Let's see if $fi = 0$. Fix an arbitrary n and let $x \in \ker(f_n)$. Then $i_n(x) = x \in B_n$, and $f_n(x) = 0$ since $x \in \ker(f_n)$. So $fi = 0$ as desired.

This is actually speaking to something in particular: the kernel as defined in this book is a map, but the kernel as we know it in other algebraic settings is a subobject. How can we reconcile the two? Does the universal property in some sense make the domain of the kernel map unique? Good questions to ask.

Our approach to working the kernel half of the problem seemed to go well, so let's do the cokernel in the same manner. Again have $f : B_\bullet \to C_\bullet$. Then $\mathrm{coker}(f)$ is the subcomplex of C

$$\cdots \to \mathrm{coker}(f_{n+1}) \xrightarrow{\partial|_{\mathrm{coker}(f_{n+1})}} \mathrm{coker}(f_n) \xrightarrow{\partial|_{\mathrm{coker}(f_n)}} \mathrm{coker}(f_{n-1}) \to \cdots .$$

There is a natural map $B_\bullet \xrightarrow{f} C_\bullet \xrightarrow{e} \mathrm{coker}(f)$ given by restriction. So we show $ef = 0$. Fix n; let $x \in B_n$. Then $f_n(x) \in C_n$, and $e_n(f_n(x)) = 0 \in \mathrm{coker}(f_n)$.

I also am worried about the universal properties. Is it a thing you should check?

Definition 1.2.2 An *abelian category* is an additive category $\mathcal{A}$ such that

1. every map in $\mathcal{A}$ has a kernel and cokernel.
2. every monic in $\mathcal{A}$ is the kernel of its cokernel.
3. every epi in $\mathcal{A}$ is the cokernel of its kernel.

The prototype abelian category is the category **mod**-R of R-modules. In any abelian category the *image* $\mathrm{im}(f)$ of a map $f : B \to C$ is the subobject $\ker(\mathrm{coker}\, f)$ of C; in the category of R-modules, $\mathrm{im}(f) = \{f(b) \mid b \in B\}$. Every map f factors as

$$B \xrightarrow{e} \mathrm{im}(f) \xrightarrow{m} C$$

with e an epimorphism and m a monomorphism. A sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of maps in $\mathcal{A}$ is called *exact* (at B) if $\ker(g) = \mathrm{im}(f)$.

A subcategory $\mathcal{B}$ of $\mathcal{A}$ is called an *abelian subcategory* if it is abelian, and an exact sequence in $\mathcal{B}$ is also exact in $\mathcal{A}$.

If $\mathcal{A}$ is any abelian category, we can repeat the discussion of section 1.1 to define chain complexes and chain maps in $\mathcal{A}$-just replace **mod**-R by $\mathcal{A}$! These form an additive category $\mathbf{Ch}(\mathcal{A})$, and homology becomes a functor from this category to $\mathcal{A}$. In the sequel we will merely write **Ch** for $\mathbf{Ch}(\mathcal{A})$ when $\mathcal{A}$ is understood.

Theorem 1.2.3 *The category $\mathbf{Ch} = \mathbf{Ch}(\mathcal{A})$ of chain complexes is an abelian category.*

Proof. Condition 1 was exercise 1.2.3 above. If $f : B \to C$ is a chain map, I claim that f is monic if and only if each $B_n \to C_n$ is monic, that is, B is isomorphic to a subcomplex of C. This follows from the fact that the composite $\ker(f) \to C$ is zero, so if f is monic, then $\ker(f) = 0$. So if f is monic, it is isomorphic to the kernel of $C \to {}^{C}\!/\!_{B}$. Similarly, f is an epi if and only if each $B_n \to C_n$ is an epi, that is, C is isomorphic to the cokernel of the chain map $\ker(f) \to B$. ☐

Exercise 1.2.4 Show that a sequence $0 \to A \to B \to C \to 0$ of chain complexes is exact in **Ch** just in case each sequence $0 \to A_n \to B_n \to C_n \to 0$ is exact in $\mathcal{A}$.

"Just in case"?? I'm assuming perhaps "if and only if," and we'll see if we run into roadbloacks in either direction.

Let $0 \to A_n \to B_n \to C_n \to 0$ be exact in $\mathcal{A}$ for every n. Then for all n, $0 = \ker(A_n \to B_n)$, $\operatorname{im}(A_n \to B_n) = \ker(B_n \to C_n)$, and $\operatorname{im}(B_n \to C_n) = C_n$. By **Exercise 1.2.3**, this is the case if and only if $0 = \ker(A \to B)$, $\operatorname{im}(A \to B) = \ker(B \to C)$, and $\operatorname{im}(B \to C) = C$.

Is it really that easy? I feel like I'm missing something.

Clearly we can iterate this construction and talk about chain complexes of chain complexes; these are usually called double complexes.

Example 1.2.5 A *double complex* (or *bicomplex*) in $\mathcal{A}$ is a family $\{C_{p,q}\}$ of objects of $\mathcal{A}$, together with maps

$$d^h : C_{p,q} \to C_{p-1,q} \qquad \text{and} \qquad d^v : C_{p,q} \to C_{p,q-1}$$

such that $d^h \circ d^h = d^v \circ d^v = d^v d^h + d^h d^v = 0$. It is useful to picture the bicomplex $C_{\bullet\bullet}$ as a lattice

$$\begin{tikzcd}
 & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \\
\cdots & C_{p-1,q+1} \arrow[l] \arrow[d, "d^v"'] & C_{p,q+1} \arrow[l, "d^h"'] \arrow[d, "d^v"] & C_{p+1,q+1} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & \cdots \arrow[l] \\
\cdots & C_{p-1,q} \arrow[l] \arrow[d, "d^v"'] & C_{p,q} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & C_{p+1,q} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & \cdots \arrow[l] \\
\cdots & C_{p-1,q-1} \arrow[l] \arrow[d] & C_{p,q-1} \arrow[l, "d^h"'] \arrow[d] & C_{p+1,q-1} \arrow[l, "d^h"'] \arrow[d] & \cdots \arrow[l] \\
 & \cdots & \cdots & \cdots &
\end{tikzcd}$$

in which the maps d^h go horizontally, the maps d^v go vertically, and each square anticommutes. Each row $C_{*,q}$ and each column $C_{p,*}$ is a chain complex.

We say that a double complex C is *bounded* if C has only finitely man nonzero terms along each diagonal line $p + q = n$, for example, if C is concentrated in the first quadrant of the plane (a *first quadrant double complex*).

Sign Trick 1.2.5 Because of the anticommutivity, the maps d^v are not maps in **Ch**, but chain maps $f_{*,q}$ from $C_{*,q}$ to $C_{*,q-1}$ can be defined by introducts $\pm$ signs:

$$f_{p,q} = (-1)^p d^v_{p,q} : C_{p,q} \to C_{p,q-1}.$$

Using this sign trick, we can identify the category of double complexes with the category **Ch**(**Ch**) of chain complexes in the abelian category **Ch**.

Total Complexes 1.2.6 To see why the anticommutative condition $d^v d^h + d^h d^v = 0$ is useful, define the *total complexes* $\operatorname{Tot}(C) = \operatorname{Tot}^{\prod}(C)$ and $\operatorname{Tot}^{\oplus}(C)$ by

$$\operatorname{Tot}^{\prod}(C)_n = \prod_{p+q=n} C_{p,q} \qquad \text{and} \qquad \operatorname{Tot}^{\oplus}(C)_n = \bigoplus_{p+q=n} C_{p,q}.$$

The formula $d = d^h + d^v$ defines maps (check this!)

$$d : \mathrm{Tot}^{\Pi}(C)_n \to \mathrm{Tot}^{\Pi}(C)_{n-1} \qquad \text{and} \qquad d : \mathrm{Tot}^{\oplus}(C)_n \to \mathrm{Tot}^{\oplus}(C)_{n-1}$$

such that $d \circ d = 0$, making $\mathrm{Tot}^{\Pi}(C)$ and $\mathrm{Tot}^{\oplus}(C)$ into chain complexes. Note that $\mathrm{Tot}^{\oplus}(C) = \mathrm{Tot}^{\Pi}(C)$ if C is bounded, and especially if C is a first quadrant double complex. The difference between $\mathrm{Tot}^{\Pi}(C)$ and $\mathrm{Tot}^{\oplus}(C)$ will become apparent in Chapter 5 when we discuss spectral sequences.

Remark $\mathrm{Tot}^{\Pi}(C)$ and $\mathrm{Tot}^{\oplus}(C)$ do not exist in all abelian categories; they don't exist when $\mathcal{A}$ is the category of all finite abelian groups. We say that an abelian category is *complete* if all infinite direct products exist (and so Tot^{Π} exists) and that is is *cocomplete* if all infinite direct sums exist (and so $\mathrm{Tot}^{\oplus}$ exists). Both these axioms hold in R-**mod** and in the category of chain complexes of R-modules.

Exercise 1.2.5 Give an elementary proof that $\mathrm{Tot}(C)$ is acyclic whenever C is a bounded double complex with exact rows (or exact columns). We will see later that this result follows from the Acyclic Assembly Lemma 2.7.3. It also follows from a spectral sequence argument (see Definition 5.6.2 and exercise 5.6.4).

For the sake of concreteness, C will be first quadrant, and likely R-modules. Any arguments should be pretty clear by the 3×3 case, so we only do that, and leave the generalizations up to the intrepid fool. Picture:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & C_{1,3} & \longleftarrow & C_{2,3} & \longleftarrow & C_{3,3} & \longleftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & C_{1,2} & \longleftarrow & C_{2,2} & \longleftarrow & C_{3,2} & \longleftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & C_{1,1} & \longleftarrow & C_{2,1} & \longleftarrow & C_{3,1} & \longleftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Start at $C_{1,1}$. Let $c_{1,1} \in C_{1,1} = \ker(d)$. We need to show there exists $c_{1,2} + c_{2,1} \in C_{1,2} \oplus C_{2,1}$ such that $d(c_{1,2} + c_{2,1}) = d^v(c_{1,2}) + d^h(c_{2,1}) = c_{1,1}$. As the rows are exact, $\ker(d^h) = \mathrm{im}(d^h)$, so since $d^h(c_{1,1}) = 0$, we can choose $c_{2,1} \in C_{2,1}$ such that $d^h(c_{2,1}) = c_{1,1}$. We just need $c_{1,2} \in C_{1,2}$, but just choose $0 \in C_{1,2}$. Then $d(0 + c_{2,1}) = d^v(0) + d^h(c_{2,1}) = 0 + c_{1,1}$, so $c_{1,1} \in \mathrm{im}(d)$, as desired.

•••

Now let $c_{1,2} + c_{2,1} \in \ker(d) \subseteq C_{1,2} \oplus C_{2,1}$. We need to show there exists $c_{1,3} + c_{2,2} + c_{3,1} \in C_{1,3} \oplus C_{2,2} \oplus C_{3,1}$ such that $d(c_{1,3} + c_{2,2} + c_{3,1}) = c_{1,2} + c_{2,1}$. As before, take $c_{1,3} \in C_{1,3}$ to be 0. We continue to write $c_{1,3}$ for now just so that the process is clearer when we generalize. Now compute

$$d^h(c_{1,2} - d^v c_{1,3}) = d^h c_{1,2} - d^h d^v c_{1,3} = d^h c_{1,2} + d^v d^h c_{1,3} = d^h c_{1,2} + d^v 0 = 0 + 0,$$

so $c_{1,2} - d^v c_{1,3} \in \ker(d^h) = \operatorname{im}(d^h)$, so there exists $c_{2,2} \in C_{2,2}$ such that $d^h c_{2,2} = c_{1,2} - d^v c_{1,3}$. Then $c_{1,2} = d^h c_{2,2} + d^v c_{1,3}$.

The idea repeats: We have $c_{2,2}$, and we compute

$$\begin{aligned} d^h(c_{2,1} - d^v c_{2,2}) &= d^h c_{2,1} - d^h d^v c_{2,2} \\ &= d^h c_{2,1} + d^v d^h c_{2,2} \\ &= d^h c_{2,1} + d^v(c_{1,2} - d^v c_{1,3}) \\ &= d^h c_{2,1} + d^v c_{1,2} - d^v d^v c_{1,3} \end{aligned}$$

$$\begin{aligned} &= d^h c_{2,1} + d^v c_{1,2} \\ &= 0 + 0, \end{aligned}$$

so $c_{2,1} - d^v c_{2,2} \in \ker(d^h) = \operatorname{im}(d^h)$, so there exists $c_{3,1} \in C_{3,1}$ such that $d^h c_{3,1} = c_{2,1} - d^v c_{2,2}$. Then $c_{2,1} = d^h c_{3,1} + d^v c_{2,2}$.

This completes the process. We have $c_{1,3} + c_{2,2} + c_{3,1} \in C_{1,3} \oplus C_{2,2} \oplus C_{3,1}$, and see that

$$\begin{aligned} d(c_{1,3} + c_{2,2} + c_{3,1}) &= d^h(c_{1,3} + c_{2,2} + c_{3,1}) + d^v(c_{1,3} + c_{2,2} + c_{3,1}) \\ &= d^h c_{1,3} + (d^h c_{2,2} + d^v c_{1,3}) + (d^h c_{3,1} + d^v c_{2,2}) + d^v c_{3,1} \\ &= 0 + c_{1,2} + c_{2,1} + 0, \end{aligned}$$

as we wished to show.

This process generalizes; take the top left corner element to be zero and use exactness of rows to work down the diagonal. In fact, this means I *don't* want to think about the 3×3 case; I just want to use the fact that the double complex is bounded to use our described process. Work down the diagonal from top left to bottom right, and as the complex is bounded, your process terminates. I am the intrepid fool.

Exercise 1.2.6 Give examples of (1) a second quadrant double complex C with exact columns such that $\mathrm{Tot}^{\Pi}(C)$ is acyclic but $\mathrm{Tot}^{\oplus}(C)$ is not; (2) a second quadrant double complex C with exact rows such that $\mathrm{Tot}^{\oplus}(C)$ is acyclic but $\mathrm{Tot}^{\Pi}(C)$ is not; and (3) a double complex (in the entire plane) for which every row and every column is exact, yet neither $\mathrm{Tot}^{\Pi}(C)$ nor $\mathrm{Tot}^{\oplus}(C)$ is acyclic.

1. Consider

2.

3.

Truncation 1.2.7 If C is a chain complex and n is an integer, we let $\tau_{\geq n}C$ denote the subcomplex of C defined by

$$(\tau_{\geq n}C)_i = \begin{cases} 0 & \text{if } i < n \\ Z_n & \text{if } i = n \\ C_i & \text{if } i > n. \end{cases}$$

Clearly $H_i(\tau_{\geq n}C) = 0$ for $i < n$ and $H_i(\tau_{\geq n}C) = H_i(C)$ for $i \geq n$. The complex $\tau_{\geq n}C$ is called the (good) *truncation* of C below n, and the quotient complex $\tau_{<n}C = C/(\tau_{\geq n}C)$ is called the (good) truncation of C above n; $H_i(\tau_{<n}C)$ is $H_i(C)$ for $i < n$ and 0 for $i \geq n$.

Some less useful variants are the *brutal truncations* $\sigma_{<n}C$ and $\sigma_{\geq n}C = C/(\sigma_{<n}C)$. By definition, $(\sigma_{<n}C)_i$ is C_i if $i < n$ and 0 if $i \geq n$. These have the advantage of being easier to describe but the disadvantage of introducing the homology group $H_n(\sigma_{\geq n}C) = C_n/B_n$.

Translation 1.2.8 Shifting indices, or translation, is another useful operation we can perform on chain and cochain complexes. If C is a complex and p an integer, we form a new complex $C[p]$ as follows:

$$C[p]_n = C_{n+p} \text{ (resp. } C[p]^n = C^{n-p})$$

with differential $(-1)^p d$. We call $C[p]$ the p^{th} *translate* of C. The way to remember the shift is that the degree 0 part of $C[p]$ is C_p. The sign convention is designed to simplify notation later on. Note that translation shifts homology:

$$H_n(C[p]) = H_{n+p}(C) \text{ (resp. } H^n(C[p]) = H^{n-p}(C)).$$

We make translation a functor by shifting indices on chain maps. That is, if $f : C \to D$ is a chain map, then $f[p]$ is the chain map given by the formula

$$f[p]_n = f_{n+p} \text{ (resp. } f[p]^n = f^{n-p}).$$

Exercise 1.2.7 If C is a complex, show that there are exact sequences of complexes:

$$0 \to Z(C) \to C \xrightarrow{d} B(C)[-1] \to 0;$$

$$0 \to H(C) \to C/B(C) \xrightarrow{d} Z(C)[-1] \to H(C)[-1] \to 0.$$

Exercise 1.2.8 (Mapping cone) Let $f : B \to C$ be a morphism of chain complexes. Form a double chain complex D out of f by thinking of f as a chain complex in **Ch** and using the sign trick, putting $B[-1]$ in the row $q = 1$ and C in the row $q = 0$. Thinking of C and $B[-1]$ as double complexes in the obvious way, show that there is a short exact sequence of double complexes

$$0 \to C \to D \xrightarrow{\delta} B[-1] \to 0.$$

The total complex of D is cone(f'), the mapping cone (see section 1.5) of a map f', which differs from f only by some $\pm$ signs and is isomorphic to f.

Shifting B in the construction of D is an error; then there's no way to make the vertical maps in D. Also, $\mathrm{Tot}(D)_n$ needs to be $B_{n-1} \oplus C_n$ as per section 1.5. The double complex D is

$$\begin{tikzcd}
0 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \\
\cdots & B_{n-1} \arrow[l, "-d^D_{n-1}"'] \arrow[d, "(-1)^{n-1} f_{n-1}"] & B_n \arrow[l, "-d^D_n"'] \arrow[d, "(-1)^n f_n"] & B_{n+1} \arrow[l, "-d^D_{n+1}"'] \arrow[d, "(-1)^{n+1} f_{n+1}"] & \cdots \arrow[l, "-d^D_{n+1}"'] \\
\cdots & C_{n-1} \arrow[l, "d^C_{n-1}"] \arrow[d] & C_n \arrow[l, "d^C_n"] \arrow[d] & C_{n+1} \arrow[l, "d^C_{n+1}"] \arrow[d] & \cdots \arrow[l, "d^C_{n+2}"'] \\
0 & 0 & 0 & 0 & 0
\end{tikzcd}$$

The exactness of $0 \to C \to D \to B[-1] \to 0$ is obvious; C includes into D and D projects onto B, then run the differential to get $B[-1]$. Running two maps obviously hits 0.

1.3 Long Exact Sequences

It is time to unveil the feature that makes chain complexes so special from a computational viewpoint: the existence of long exact sequences.

Theorem 1.3.1 *Let $0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0$ be a short exact sequence of chain complexes. Then there are natural maps $\partial : H_n(C) \to H_{n-1}(A)$, called connecting homomorphisms, such that*

$$\cdots \xrightarrow{g} H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{f} H_n(B) \xrightarrow{g} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{f} \cdots$$

is an exact sequence.

Similarly, if $0 \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \to 0$ is a short exact sequence of cochain complexes, there are natural maps $\partial : H^n(C) \to H^{n+1}(A)$ and a long exact sequence

$$\cdots \xrightarrow{g} H^{n-1}(C) \xrightarrow{\partial} H^n(A) \xrightarrow{f} H^n(B) \xrightarrow{g} H^n(C) \xrightarrow{\partial} H^{n+1}(A) \xrightarrow{f} \cdots.$$

Exercise 1.3.1 Let $0 \to A \to B \to C \to 0$ be a short exact sequence of complexes. Show that if two of the three complexes A, B, C are exact, then so is the third.

Write $0 \to A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \to 0$, and assume δ is the differential on A, ∂ on B, and d on C.

First, assume A and B are exact. We need to show $\ker(d_n) = \operatorname{im}(d_{n+1})$. We have

$$\begin{tikzcd}
0 \arrow[r] & A_{n+1} \arrow[r] \arrow[d] & B_{n+1} \arrow[r] \arrow[d] & C_{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] & B_{n-1} \arrow[r] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

Since $\operatorname{im}(d_{n+1}) \subseteq \ker(d_n)$ always, we show the other inclusion. Let $c_n \in \ker(d_n)$. Then, since $B_n \to C_n \to 0$ is exact, there exists some $b_n \in B_n$ such that $b_n \xmapsto{g_n} c_n$. Now focus on the square

$$\begin{tikzcd}
B_n \arrow[r, "g_n"] \arrow[d, "\partial_n"'] & C_n \arrow[d, "d_n"] \\
B_{n-1} \arrow[r, "g_{n-1}"'] & C_{n-1}
\end{tikzcd},$$

which we know commutes. So that means $g_{n-1}\partial_n b_n = d_n g_n b_n = d_n c_n = 0$, and therefore $\partial_n b_n \in \ker(g_{n-1})$, which by exactness is $\operatorname{im}(f_{n-1})$, so there exists $a_{n-1} \in A_{n-1}$ such that $f_{n-1}a_{n-1} = \partial_n b_n$. Now, focus on the piece

$$\begin{tikzcd}
0 \arrow[r] & A_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "\delta_{n-1}"'] & B_{n-1} \arrow[d, "\partial_{n-1}"] \\
0 \arrow[r] & A_{n-2} \arrow[r, "f_{n-2}"'] & B_{n-2}
\end{tikzcd}.$$

Again, by commutativity, $f_{n-2}\delta_{n-1}a_{n-1} = \partial_{n-1}f_{n-1}a_{n-1} = \partial_{n-1}\partial_n b_n = 0$ as ∂ is a differential. Thus, $f_{n-2}\delta_{n-1}a_{n-1} = 0$, but since $0 \to A_{n-2} \xrightarrow{f_{n-2}} B_{n-2}$ is exact, f_{n-2} is injective, so $\delta_{n-1}a_{n-1} = 0$, and thus $a_{n-1} \in \ker(\delta_{n-1}) = \operatorname{im}(\delta_n)$. So there exists $a_n \in A_n$ such that $\delta_n(a_n) = a_{n-1}$. Now, we are here:

$$\begin{tikzcd}
A_n \arrow[r, "f_n"] \arrow[d, "\delta_n"'] & B_n \arrow[d, "\partial_n"] \\
A_{n-1} \arrow[r, "f_{n-1}"'] & B_{n-1}
\end{tikzcd}.$$

Consider $b_n - f_n a_n \in B_n$. See that

$$\partial_n(b_n - f_n a_n) = \partial_n b_n - \partial_n f_n a_n = \partial_n b_n - f_{n-1}\delta_n a_n = \partial_n b_n - f_{n-1}a_{n-1} = \partial_n b_n - \partial_n b_n = 0,$$

so $b_n - f_n a_n \in \ker \partial_n = \operatorname{im} \partial_{n+1}$. Thus there is $b_{n+1} \in B_{n+1}$ such that $\partial_{n+1}b_{n+1} = b_n - f_n a_n$. Finally, look here:

$$
\begin{tikzcd}
B_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "\partial_{n+1}"'] & C_{n+1} \arrow[d, "d_{n+1}"] \\
B_n \arrow[r, "g_n"'] & C_n
\end{tikzcd}.
$$

See that

$$
d_{n+1}g_{n+1}b_{n+1} = g_n\partial_{n+1}b_{n+1} = g_n(b_n - f_na_n) = g_nb_n - g_nf_na_n = c_n - 0 = c_n,
$$

so take $c_{n+1} = g_{n+1}b_{n+1}$; then $d_{n+1}c_{n+1} = c_n$, and $c_n \in \operatorname{im}(d_{n+1})$, as desired. n arbitrary makes $C_\bullet$ exact.

•••

Now, assume A and C are exact. We need to show $\ker(\partial_n) = \operatorname{im}(\partial_{n+1})$. Again,

$$
\begin{tikzcd}
0 \arrow[r] & A_{n+1} \arrow[r] \arrow[d] & B_{n+1} \arrow[r] \arrow[d] & C_{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] & B_{n-1} \arrow[r] & C_{n-1} \arrow[r] & 0
\end{tikzcd},
$$

and again, it is enough to show that $\ker(\partial_n) \subseteq \operatorname{im}(\partial_{n+1})$. Let $b_n \in \ker(\partial_n)$. Then $\partial_nb_n = 0$. Consider

$$
\begin{tikzcd}
B_n \arrow[r, "g_n"] \arrow[d, "\partial_n"'] & C_n \arrow[r] \arrow[d, "d_n"] & 0 \\
B_{n-1} \arrow[r, "g_{n-1}"'] & C_{n-1} \arrow[r] & 0
\end{tikzcd}.
$$

By commutativity, $d_ng_nb_n = g_{n-1}\partial_nb_n = g_{n-1}0 = 0$, so $g_nb_n \in \ker(d_n) = \operatorname{im}(d_{n+1})$, so there exists $c_{n+1} \in C_{n+1}$ such that $d_{n+1}c_{n+1} = g_nb_n$. Since $B_{n+1} \xrightarrow{g_{n+1}} C_{n+1} \to 0$ is exact, there exists $b_{n+1} \in B_{n+1}$ such that $g_{n+1}b_{n+1} = c_{n+1}$. Move up a square:

$$
\begin{tikzcd}
B_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "\partial_{n+1}"'] & C_{n+1} \arrow[d, "d_{n+1}"] \\
B_n \arrow[r, "g_n"'] & C_n
\end{tikzcd},
$$

and consider $b_n - \partial_{n+1}b_{n+1} \in B_n$. See that

$$\begin{aligned}
g_n(b_n - \partial_{n+1}b_{n+1}) &= g_nb_n - g_n\partial_{n+1}b_{n+1} \\
&= g_nb_n - d_{n+1}g_{n+1}b_{n+1} \\
&= g_nb_n - d_{n+1}c_{n+1} \\
&= g_nb_n - g_nb_n \\
&= 0,
\end{aligned}$$

so $b_n - \partial_{n+1}b_{n+1} \in \ker(g_n) = \operatorname{im}(f_n)$. Thus there exists $a_n \in A_n$ such that $f_na_n = b_n - \partial_{n+1}b_{n+1}$. On the piece

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r, "f_n"] \arrow[d, "\delta_n"'] & B_n \arrow[d, "\partial_n"] \\
0 \arrow[r] & A_{n-1} \arrow[r, "f_{n-1}"'] & B_{n-1}
\end{tikzcd},$$

we get

$$f_{n-1}\delta_na_n = \partial_nf_na_n = \partial_n(b_n - \partial_{n+1}b_{n+1}) = \partial_nb_n - \partial_n\partial_{n+1}b_{n+1} = 0 - 0 = 0.$$

$0 \to A_{n-1} \xrightarrow{f_{n-1}} B_{n-1}$ exact makes f_{n-1} injective, so $\delta_na_n = 0$, so $a_n \in \ker\delta_n = \operatorname{im}\delta_{n+1}$, so there exists $a_{n+1} \in A_{n+1}$ such that $\delta_{n+1}a_{n+1} = a_n$. Move to square

$$\begin{tikzcd}
A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "\delta_{n+1}"'] & B_{n+1} \arrow[d, "\partial_{n+1}"] \\
A_n \arrow[r, "f_n"'] & B_n
\end{tikzcd}.$$

Here, consider $b_{n+1} + f_{n+1}a_{n+1}$ in B_{n+1}. We claim that this maps to b_n under ∂_{n+1}, and hence $b_n \in \operatorname{im}(\partial_{n+1})$, completing the proof. So see

$$\begin{aligned}
\partial_{n+1}(b_{n+1} + f_{n+1}a_{n+1}) &= \partial_{n+1}b_{n+1} + \partial_{n+1}f_{n+1}a_{n+1} \\
&= \partial_{n+1}b_{n+1} + f_n\delta_{n+1}a_{n+1} \\
&= \partial_{n+1}b_{n+1} + f_na_n \\
&= \partial_{n+1}b_{n+1} + b_n - \partial_{n+1}b_{n+1} \\
&= b_n,
\end{aligned}$$

as claimed.

•••

Now, assume B and C are exact. We need to show $\ker(\delta_n) = \operatorname{im}(\delta_{n+1})$. Once more,

$$\begin{tikzcd}
0 \arrow[r] & A_{n+1} \arrow[r] \arrow[d] & B_{n+1} \arrow[r] \arrow[d] & C_{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] & B_{n-1} \arrow[r] & C_{n-1} \arrow[r] & 0
\end{tikzcd},$$

and we show that $\ker(\delta_n) \subseteq \operatorname{im}(\delta_{n+1})$. Let $a_n \in \ker(\delta_n)$. Then $\delta_n a_n = 0$. Consider

$$\begin{tikzcd}
A_n \arrow[r, "f_n"] \arrow[d, "\delta_n"'] & B_n \arrow[d, "\partial_n"] \\
A_{n-1} \arrow[r, "f_{n-1}"'] & B_{n-1}
\end{tikzcd}.$$

The diagram commutes, so $\partial_n f_n a_n = f_{n-1}\delta_n a_n = f_{n-1}0 = 0$, so $f_n a_n \in \ker \partial_n = \operatorname{im} \partial_{n+1}$. So there exists $b_{n+1} \in B_{n+1}$ such that $\partial_{n+1} b_{n+1} = f_n a_n$. Now go here:

$$\begin{tikzcd}
B_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "\partial_{n+1}"'] & C_{n+1} \arrow[d, "d_{n+1}"] \\
B_n \arrow[r, "g_n"'] & C_n
\end{tikzcd}.$$

We have $d_{n+1} g_{n+1} b_{n+1} = g_n \partial_{n+1} b_{n+1} = g_n f_n a_n = 0$, so $g_{n+1} b_{n+1} \in \ker(d_{n+1}) = \operatorname{im}(d_{n+2})$. Thus there exists $c_{n+2} \in C_{n+2}$ such that $d_{n+2} c_{n+2} = g_{n+1} b_{n+1}$. By the exactness of the sequence $B_{n+2} \xrightarrow{g_{n+2}} C_{n+2} \to 0$, there is some $b_{n+2} \in B_{n+2}$ such that $g_{n+2} b_{n+2} = c_{n+2}$. Now at the square

$$\begin{tikzcd}
B_{n+2} \arrow[r, "g_{n+2}"] \arrow[d, "\partial_{n+2}"'] & C_{n+2} \arrow[d, "d_{n+2}"] \\
B_{n+1} \arrow[r, "g_{n+1}"'] & C_{n+1}
\end{tikzcd},$$

we get $g_{n+1} \partial_{n+2} b_{n+2} = d_{n+2} g_{n+2} b_{n+2} = d_{n+2} c_{n+2} = g_{n+1} b_{n+1}$, so $g_{n+1}(\partial_{n+2} b_{n+2} - b_{n+1}) = 0$, and thus $\partial_{n+2} b_{n+2} - b_{n+1} \in \ker(g_{n+1}) = \operatorname{im}(f_{n+1})$. Thus there exists $a_{n+1} \in A_{n+1}$ such that $f_{n+1} a_{n+1} = \partial_{n+2} b_{n+2} - b_{n+1}$. Finally, the square

$$
\begin{tikzcd}
A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "\delta_{n+1}"'] & B_{n+1} \arrow[d, "\partial_{n+1}"] \\
A_n \arrow[r, "f_n"'] & B_n
\end{tikzcd}
$$

commuting means that

$$
\begin{aligned}
f_n \delta_{n+1} a_{n+1} &= \partial_{n+1} f_{n+1} a_{n+1} \\
&= \partial_{n+1}(\partial_{n+2} b_{n+2} - b_{n+1}) \\
&= \partial_{n+1}\partial_{n+2} b_{n+2} - \partial_{n+1} b_{n+1} \\
&= 0 - f_n a_n.
\end{aligned}
$$

So $f_n(\delta_{n+1}(-a_{n+1})) = f_n(a_n)$, and since $0 \to A_n \xrightarrow{f_n} B_n$ is exact, f_n is injective, and thus $\delta_{n+1}(-a_{n+1}) = a_n$. Therefore $a_n \in \operatorname{im} \delta_{n+1}$, as we wished to show.

Exercise 1.3.2 (3×3 lemma) Suppose given a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

in an abelian category, such that every column is exact. Show the following:

1. If the bottom two rows are exact, so is the top row.
2. If the top two rows are exact, so is the bottom row.
3. If the top and bottom rows are exact, and the composite $A \to C$ is zero, the middle row is also exact.

Hint: Show the remaining row is a complex, and apply exercise 1.3.1.

1. Suppose the bottom two rows are exact. We need to show that $0 \to A' \xrightarrow{d_1{}'} B' \xrightarrow{d_2{}'} C' \to 0$ is a complex; i.e., that $d_2{}' \circ d_1{}' = 0$. Let $a' \in A'$ and we compute $d_2{}'d_1{}'a'$. Since the diagram commutes,

$$\begin{tikzcd}
A' \arrow[r, "d_1{}'"] \arrow[d, "\alpha"'] & B' \arrow[d, "\beta"] \\
A \arrow[r, "d_1"'] & B
\end{tikzcd},$$

$d_1\alpha a' = \beta d_1{}' a'$

2.

3. $A \to C$ zero automatically means that the middle row is a complex. Apply exercise 1.3.1.

The key tool in constructing the connecting homomorphism ∂ is our next result, the *Snake Lemma.* We will not print the proof in these notes, because it is best done visually. In fact, a clear proof is given by Jill Clayburgh at the beginning of the movie *It's My Turn* (Rastar-Martin Elfand Studios, 1980). As an exercise in "diagram chasing" of elements, the student should find a proof (but privately - keep the proof to yourself!).

Snake Lemma 1.3.2 *Consider a commutative diagram of R-modules of the form*

$$\begin{tikzcd}
 & A' \arrow[r] \arrow[d, "f"] & B' \arrow[r, "p"] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r] & C &
\end{tikzcd}.$$

If the rows are exact, there is an exact sequence

$$\ker(f) \to \ker(g) \to \ker(h) \xrightarrow{\partial} \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h)$$

with ∂ defined by the formula

$$\partial(c') = i^{-1}gp^{-1}(c'), \qquad\qquad c' \in \ker(h).$$

Moreover if $A' \to B'$ is monic, then so is $\ker(f) \to \ker(g)$*, and if $B \to C$ is onto, then so is* $\operatorname{coker}(g) \to \operatorname{coker}(h)$*.*

Etymology The term *snake* comes from the following visual mnemonic:

$$\begin{tikzcd}
\ker(f) \arrow[r] \arrow[d] & \ker(g) \arrow[r] \arrow[d] & \ker(h) \arrow[d] \arrow[dddll, dashed] \\
\bullet \arrow[r] \arrow[d, "f"'] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d, "h"] \\
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\operatorname{coker}(f) \arrow[r] & \operatorname{coker}(g) \arrow[r] & \operatorname{coker}(h)
\end{tikzcd}$$

Remark The Snake Lemma also holds in an arbitrary abelian category $\mathcal{C}$. To see this, let $\mathcal{A}$ be the smallest abelian subcategory of $\mathcal{C}$ containing the objects and morphisms of the diagram. Since $\mathcal{A}$ has a set of objects, the Freyd-Mitchell Embedding Theorem (see 1.6.1) gives an exact, fully faithful embedding of $\mathcal{A}$ into R-**mod** for some ring R. Since ∂ exists in R-**mod**, it exists in $\mathcal{A}$ and hence in $\mathcal{C}$. Similarly, exactness in R-**mod** implies exactness in $\mathcal{A}$ and hence in $\mathcal{C}$.

Exercise 1.3.3 (5-Lemma) In any commutative diagram

$$\begin{tikzcd}
A' \arrow[r] \arrow[d, "a"', "\cong"] & B' \arrow[r] \arrow[d, "b"', "\cong"] & C' \arrow[r] \arrow[d, "c"'] & D' \arrow[r] \arrow[d, "d"', "\cong"] & E' \arrow[d, "e"', "\cong"] \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & D \arrow[r] & E
\end{tikzcd}$$

with exact rows in any abelian category, show that if a, b, d, and e are isomorphisms, then c is also an isomorphism. More precisely, show that if b and d are monic and a is an epi, then c is monic. Dually, show that if b and d are epis and e is monic, then c is an epi.

Let's label maps a bit more so that we can chase diagrams. I'm slowly becoming less precise, so we'll just call ∂' the map from $\bullet' \to \bullet'$ and ∂ the map from $\bullet \to \bullet$, and know which specific map we mean by context. Maybe by the end of this book I'll be as blasé as Weibel.

First let's show that c is monic (injective). Let $\gamma' \in C'$ and assume $c\gamma' = 0$. Then

$$\begin{tikzcd}
C' \arrow[r] \arrow[d, "c"'] & D' \arrow[d, "d"] \\
C \arrow[r] & D
\end{tikzcd},$$

so $\partial c\gamma' = \partial 0 = 0$. Since d is an isomorphism and the hint points us towards its monic-ness, $\partial'\gamma' = 0$. By exactness of the top row, $\ker\partial' = \operatorname{im}\partial'$, so there exists $\beta' \in B'$ such that $\partial'\beta' = \gamma'$. Now look here:

$$\begin{tikzcd} B' \arrow[r] \arrow[d, "b"'] & C' \arrow[d, "c"] \\ B \arrow[r] & C \end{tikzcd}$$

Since the square commutes, $\partial b\beta' = c\partial'\beta' = c\gamma' = 0$, so $b\beta' \in \ker\partial = \operatorname{im}\partial$, so there exists $\alpha \in A$ such that $\partial\alpha = b\beta'$.

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "a"'] & B' \arrow[d, "b"] \\ A \arrow[r] & B \end{tikzcd}$$

As a is epi (surjective), there exists $\alpha' \in A'$ such that $a\alpha' = \alpha$. By the commutativity of the square, $b\partial'\alpha' = \partial a\alpha' = \partial\alpha = b\beta'$, so $b(\partial'\alpha' - \beta') = 0$. As b is monic, $\partial'\alpha' - \beta' = 0$, so $\partial'\alpha' = \beta'$. This means that $\gamma' = \partial'\beta' = \partial'\partial'\alpha' = 0$ as the rows are exact. Thus c is monic, as desired.

Now, we show that c is an epi, using that b and d are epi and e is monic. Let $\gamma \in C$; we need to show there exists $\widetilde{\gamma} \in C'$ such that $c\widetilde{\gamma} = \gamma$. On the square

$$\begin{tikzcd} C' \arrow[r] \arrow[d, "c"'] & D' \arrow[d, "d"] \\ C \arrow[r] & D \end{tikzcd},$$

as d is epi, there exists $\delta' \in D'$ such that $d\delta' = \partial\gamma$. By commutativity of the square

$$\begin{tikzcd} D' \arrow[r] \arrow[d, "d"'] & E' \arrow[d, "e"] \\ D \arrow[r] & E \end{tikzcd},$$

we get $e\partial'\delta' = \partial d\delta' = \partial\partial\gamma = 0$, and since e is monic, $\partial'\delta' = 0$, so $\delta' \in \ker\partial' = \operatorname{im}\partial'$, so there exists $\gamma' \in C'$ such that $\partial'\gamma' = \delta'$. Move to square

$$\begin{tikzcd} C' \arrow[r] \arrow[d, "c"'] & D' \arrow[d, "d"] \\ C \arrow[r] & D \end{tikzcd}$$

By its commutativity, $\partial c\gamma' = d\partial'\gamma' = d\delta' = \partial\gamma$, so $\partial(c\gamma' - \gamma) = 0$, so $c\gamma' - \gamma \in \ker\partial = \operatorname{im}\partial$, so there exists $\beta \in B$ such that $\partial\beta = c\gamma' - \gamma$. Consider square

$$\begin{tikzcd}
B' \arrow[r] \arrow[d, "b"'] & C' \arrow[d, "c"] \\
B \arrow[r] & C
\end{tikzcd}$$

As b is epi, there exists $\beta' \in B'$ such that $b\beta' = \beta$. By commutivaty of the square, $c\partial'\beta' = \partial b\beta' = \partial\beta = c\gamma' - \gamma$, so $\gamma = c(\gamma' - \partial'\beta')$, so let $\widetilde{\gamma} = \gamma' - \partial'\beta'$ and then $\gamma = c\widetilde{\gamma}$, as desired, and c is epi.

We now proceed to the construction of the connecting homomorphism ∂ of Theorem 1.3.1 associated to a short exact sequence

$$0 \to A \to B \to C \to 0$$

of chain complexes. From the Snake Lemma and the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Z_nA \arrow[r] \arrow[d] & Z_nB \arrow[r] \arrow[d] & Z_nC \arrow[d] & \\
0 \arrow[r] & A_n \arrow[r] \arrow[d, "d"'] & B_n \arrow[r] \arrow[d, "d"'] & C_n \arrow[r] \arrow[d, "d"'] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
 & A_{n-1}/dA_n \arrow[r] \arrow[d] & B_{n-1}/dB_n \arrow[r] \arrow[d] & C_{n-1}/dC_n \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

we see that the rows are exact in the commutative diagram

$$\begin{tikzcd}
 & A_n/dA_{n+1} \arrow[r] \arrow[d, "d"'] & B_n/dB_{n+1} \arrow[r] \arrow[d, "d"'] & C_n/dC_{n+1} \arrow[r] \arrow[d, "d"'] & 0 \\
0 \arrow[r] & Z_{n-1}(A) \arrow[r, "f"] & Z_{n-1}(B) \arrow[r, "g"] & Z_{n-1}(C) &
\end{tikzcd}.$$

The kernel of the left vertical is $H_n(A)$, and its cokernel is $H_{n-1}(A)$. Therefore the Snake Lemma yields an exact sequence

$$H_n(A) \xrightarrow{f} H_n(B) \xrightarrow{g} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C).$$

The long exact sequence 1.3.1 is obtained by pasting these sequences together.

Addendum 1.3.3 When one computes with modules, it is useful to be able to push elements around. By decoding the above proof, we obtain the following formula for the connecting homomorphism: Let $z \in H_n(C)$, and represent it by a cycle $c \in C_n$. Lift the cycle to $b \in B_n$ and apply d. The element db of B_{n-1} actually belongs to the submodule $Z_{n-1}(A)$ and represents $\partial(z) \in H_{n-1}(A)$.

We shall now explain what we mean by the naturality of ∂. There is a category $\mathcal{S}$ whose objects are short exact sequences of chain complexes (say, in an abelian category $\mathcal{C}$). Commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array} \qquad (*)$$

give the morphisms in $\mathcal{S}$ (from the top row to the bottom row). Similarly, there is a category $\mathcal{L}$ of long exact sequences in $\mathcal{C}$.

Proposition 1.3.4 *The long exact sequence is a functor from $\mathcal{S}$ to $\mathcal{L}$. That is, for every short exact sequence there is a long exact sequence, and for every map (*) of short exact sequences there is a commutative ladder diagram*

$$\begin{array}{ccccccccccc} \cdots & \xrightarrow{\partial} & H_n(A) & \longrightarrow & H_n(B) & \longrightarrow & H_n(C) & \xrightarrow{\partial} & H_{n-1}(A) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \xrightarrow{\partial} & H_n(A') & \longrightarrow & H_n(B') & \longrightarrow & H_n(C') & \xrightarrow{\partial} & H_{n-1}(A') & \longrightarrow & \cdots . \end{array}$$

Proof. All we have to do is establish the ladder diagram. Since each H_n is a functor, the left two squares commute. Using the Embedding Theorem 1.6.1, we may assume $\mathcal{C} = \textbf{mod-}R$ in order to prove that the right square commutes. Given $z \in H_n(C)$, represented by $c \in C_n$, its image $z' \in H_n(C')$ is represented by the image of c. If $b \in B_n$ lifts c, its image in ${B_n}'$ lifts c'. Therefore by 1.3.3 $\partial(z') \in H_{n-1}(A')$ is represented by the image of db, that is, by the image of a representative of $\partial(z)$, so $\partial(z')$ is the image of $\partial(z)$. $\square$

Remark 1.3.5 The data of the long exact sequence is sometimes organized into the mnemonic shape

$$\begin{array}{ccc} H_*(A) & \longrightarrow & H_*(B) \\ & {}_{\partial}\nwarrow \quad \swarrow & \\ & H_*(C) & \end{array}$$

This is called an *exact triangle* for obvious reasons. This mnemonic shape is responsible for the term "triangulated category," which we will discuss in Chapter 10. The category $\mathbf{K}$ of chain equivalence classes of complexes and maps (see exercise 1.4.5 in the next section) is an example of a triangulated category.

Exercise 1.3.4 Consider the boundaries-cycles exact sequence $0 \to Z \to C \to B[-1] \to 0$ associated to a chain complex C (exercise 1.2.7). Show that the corresponding long exact sequence of homology breaks up into short exact sequences.

The corresponding long exact sequence is, by Theorem 1.3.1,

$$\cdots \to H_{n+1}(B[-1]) \xrightarrow{\partial} H_n(Z) \to H_n(C) \to H_n(B[-1]) \xrightarrow{\partial} H_{n-1}(Z) \to \cdots .$$

Let's examine $H_*(Z)$ and $H_*(B[-1])$. See that Z is

$$\cdots \to Z_{n+1} \xrightarrow{d_{n+1}} Z_n \xrightarrow{d_n} Z_{n-1} \to \cdots ,$$

but as $Z = \ker(d)$, all maps are the zero map, and then $H_n(Z) = {}^{\ker(d_n)}\!/\!_{\mathrm{im}(d_{n+1})} = {}^{Z_n}\!/\!_{0} = Z_n$. For $B[-1]$, we have

$$\cdots \to B_n \xrightarrow{d_n} B_{n-1} \xrightarrow{d_{n-1}} B_{n-2} \to \cdots .$$

Now, as $B = \mathrm{im}(d)$ and d is a differential, all maps are the zero map, and

$$\begin{aligned} H_n(B[-1]) &= {}^{\ker(d_{n-1})}\!/\!_{\mathrm{im}(d_n)} \\ &= {}^{B_{n-1}}\!/\!_{0} \\ &= B_{n-1}. \end{aligned}$$

So the long exact sequence is

$$\begin{tikzcd} & \cdots \arrow[r] & B_{n+1} \arrow[dll, hook] \\ Z_{n+1} \arrow[r] & H_{n+1}(C) \arrow[r] & B_n \arrow[dll, hook] \\ Z_n \arrow[r] & H_n(C) \arrow[r] & B_{n-1} \arrow[dll, hook] \\ Z_{n-1} \arrow[r] & H_{n-1}(C) \arrow[r] & B_{n-2} \arrow[dll, hook] \\ Z_{n-2} \arrow[r] & \cdots & \end{tikzcd}$$

Now rewrite $H_*(C) = {}^{Z_*}\!/\!_{B_*}$:

$$\begin{tikzcd} & \cdots \arrow[r] & B_{n+1} \arrow[dll, hook] \\ Z_{n+1} \arrow[r] & {}^{Z_{n+1}}\!/\!_{B_{n+1}} \arrow[r] & B_n \arrow[dll, hook] \\ Z_n \arrow[r] & {}^{Z_n}\!/\!_{B_n} \arrow[r] & B_{n-1} \arrow[dll, hook] \\ Z_{n-1} \arrow[r] & {}^{Z_{n-1}}\!/\!_{B_{n-1}} \arrow[r] & B_{n-2} \arrow[dll, hook] \\ Z_{n-2} \arrow[r] & \cdots & \end{tikzcd}$$

Now it is evident that the way to break this up into short exact sequences is

$$0 \to B_n \xrightarrow{\partial} Z_n \to {Z_n}/{B_n} \to 0.$$

Indeed, ∂ is injective, $Z_n \to {Z_n}/{B_n}$ is surjective, and $\ker\left(Z_n \to {Z_n}/{B_n}\right) = \mathrm{im}(\partial) = B_n$.

Exercise 1.3.5 Let f be a morphism of chain complexes. Show that if $\ker(f)$ and $\mathrm{coker}(f)$ are acyclic, then f is a quasi-isomorphism. Is the converse true?

Let $f : A_\bullet \to B_\bullet$. It is always the case that the following is a short exact sequence:

$$0 \to \ker(f) \to A_\bullet \to \mathrm{im}(f) \to 0.$$

Using Theorem 1.3.1, there are natural connecting homomorphisms ∂ such that

$$\cdots \to H_{n+1}(\mathrm{im}(f)) \xrightarrow{\partial} H_n(\ker(f)) \to H_n(A) \to H_n(\mathrm{im}(f)) \xrightarrow{\partial} H_{n-1}(\ker(f)) \to \cdots$$

is long exact. Since $\ker(f)$ is acyclic, $H_*(\ker(f)) = 0$, so

$$\cdots \to H_{n+1}(\mathrm{im}(f)) \xrightarrow{\partial} 0 \to H_n(A) \to H_n(\mathrm{im}(f)) \xrightarrow{\partial} 0 \to \cdots,$$

and therefore $H_n(A) \to H_n(\mathrm{im}(f))$ is an isomorphism. Using the same trick,

$$0 \to \mathrm{im}(f) \to B_\bullet \to \mathrm{coker}(f) \to 0$$

is always short exact, so

$$\cdots \to H_{n+1}(\mathrm{coker}(f)) \xrightarrow{\partial} H_n(\mathrm{im}(f)) \to H_n(B) \to H_n(\mathrm{coker}(f)) \xrightarrow{\partial} H_{n-1}(\mathrm{im}(f)) \to \cdots,$$

and since $H_*(\mathrm{coker}(f)) = 0$,

$$\cdots \to 0 \xrightarrow{\partial} H_n(\mathrm{im}(f)) \to H_n(B) \to 0 \xrightarrow{\partial} H_{n-1}(\mathrm{im}(f)) \to \cdots,$$

and therefore $H_n(\mathrm{im}(f)) \to H_n(B)$ is an isomorphism. So $H_n(A) \to H_n(\mathrm{im}(f)) \to H_n(B)$ is an isomorphism, and therefore f is a quasi-isomorphism, as desired.

•••

The converse is not true. Take

$$\begin{tikzcd}
\cdots \arrow[r, "\mathrm{id}"] & \mathbf{Z} \arrow[r, "0"] \arrow[d] & \mathbf{Z} \arrow[r, "\mathrm{id}"] \arrow[d] & \mathbf{Z} \arrow[r, "0"] \arrow[d] & \mathbf{Z} \arrow[r, "p"] \arrow[d] & \mathbf{Z} \arrow[r] \arrow[d, two heads] & 0 \arrow[d] \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbf{Z}/p\mathbf{Z} \arrow[r] & 0
\end{tikzcd}$$

The homology of the top row is

$$\ker(\mathrm{id})/\mathrm{im}(0) = 0/0 = 0 \qquad \text{or} \qquad \ker(0)/\mathrm{im}(\mathrm{id}) = \mathbf{Z}/\mathbf{Z} = 0$$

when $n \neq 1$ and

$$\ker(0)/\mathrm{im}(p) = \mathbf{Z}/p\mathbf{Z}$$

when $n = 1$; the homology of the bottom row is obviously 0 when $n \neq 1$ and

$$\ker\left(\mathbf{Z}/p\mathbf{Z} \to 0\right)/\mathrm{im}\left(0 \to \mathbf{Z}/p\mathbf{Z}\right) = \left(\mathbf{Z}/p\mathbf{Z}\right)/0 = \mathbf{Z}/p\mathbf{Z}$$

when $n = 1$. The chain map, call it f, is a quasi-isomorphism. Yet see that $\ker(f)$ is

$$\cdots \xrightarrow{\mathrm{id}} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{\mathrm{id}} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{p} p\mathbf{Z} \to 0,$$

and then

$$H_1(\ker(f)) = \ker(p\mathbf{Z} \to 0)/\mathrm{im}\left(\mathbf{Z} \xrightarrow{p} p\mathbf{Z}\right) = p\mathbf{Z}/p\mathbf{Z} = 0,$$

ummmm

Exercise 1.3.6 Let $0 \to A \to B \to C \to 0$ be a short exact sequence of double complexes of modules. Show that there is a short exact sequence of total complexes, and conclude that if $\mathrm{Tot}(C)$ is acyclic, then $\mathrm{Tot}(A) \to \mathrm{Tot}(B)$ is a quasi-isomorphism.

If $0 \to A \to B \to C \to 0$ is short exact, then for all p and q, $0 \to A_{p,q} \to B_{p,q} \to C_{p,q} \to 0$ is short exact, so since $\mathrm{Tot}(X) = \prod_{p+q=n} X_{p,q}$, we get that $0 \to \mathrm{Tot}(A) \to \mathrm{Tot}(B) \to \mathrm{Tot}(C) \to 0$ is exact. The short exact sequence $0 \to \mathrm{Tot}(A) \to \mathrm{Tot}(B) \to \mathrm{Tot}(C) \to 0$ gives rise to the long exact sequence

$$\cdots \to H_{n+1}(\mathrm{Tot}(C)) \xrightarrow{\partial} H_n(\mathrm{Tot}(A)) \to H_n(\mathrm{Tot}(B)) \to H_n(\mathrm{Tot}(C)) \xrightarrow{\partial} H_{n-1}(\mathrm{Tot}(A)) \to \cdots .$$

As $H_n(\mathrm{Tot}(C)) = 0$, we get

$$\cdots \to 0 \xrightarrow{\partial} H_n(\mathrm{Tot}(A)) \to H_n(\mathrm{Tot}(B)) \to 0 \xrightarrow{\partial} H_{n-1}(\mathrm{Tot}(A)) \to \cdots ,$$

and hence $H_n(\mathrm{Tot}(A)) \to H_n(\mathrm{Tot}(B))$ is an isomorphism, as desired.

1.4 Chain Homotopies

The ideas in this section and the next are motivated by homotopy theory in topology. We begin with a discussion of a special case of historical importance. If C is any chain complex of vector spaces over a field, we can always choose vector space decompositions:

$$\begin{aligned} C_n &= Z_n \oplus B'_n, & B'_n &\cong C_n/Z_n = d(C_n) = B_{n-1}; \\ Z_n &= B_n \oplus H'_n, & H'_n &\cong Z_n/B_n = H_n(C). \end{aligned}$$

Therefore we can form the compositions

$$C_n \to Z_n \to B_n \cong B'_{n+1} \subseteq C_{n+1}$$

to get splitting maps $s_n : C_n \to C_{n+1}$, such that $d = dsd$. The compositions ds and sd are projections from C_n onto B_n and B'_n, respectively, so the sum $ds + sd$ is an endomorphism of C_n whose kernel H'_n is isomorphic to the homology $H_n(C)$. The kernel (and cokernel!) of $ds + sd$ is the trivial homology complex $H_*(C)$. Evidently both chain maps $H_*(C) \to C$ and $C \to H_*(C)$ are quasi-isomorphisms. Moreover, C is an exact sequence if and only if $ds + sd$ is the identity map.

Over an arbitrary ring R, it is not always possible to split chain complexes like this, so we give a name to this notion.

Definition 1.4.1 *A complex C is called* split *if there are maps $s_n : C_n \to C_{n+1}$ such that $d = dsd$. The maps s_n are called the* splitting maps*. If in addition C is acyclic (exact as a sequence), we say that C is* split exact*.*

Example 1.4.2 Let $R = \mathbf{Z}$ or $\mathbf{Z}/4$, and let C be the complex

$$\cdots \xrightarrow{2} \mathbf{Z}/4 \xrightarrow{2} \mathbf{Z}/4 \xrightarrow{2} \mathbf{Z}/4 \xrightarrow{2} \cdots .$$

This complex is acyclic but not split exact. There is no map s such that $ds + sd$ is the identity map, nor is there any direct sum decomposition $C_n \cong Z_n \oplus B'_n$.

Exercise 1.4.1 The previous example shows that even an acyclic chain complex of free R-modules need not be split exact.

1. Show that acyclic *bounded below* chain complexes of free R-modules are always split exact.
2. Show that an acyclic chain complex of finitely generated free abelian groups is always split

exact, even when it is not bounded below.

1. Without loss of generality, assume $C_n = 0$ for all $n \leq 0$, so

$$\cdots \xrightarrow{d_4} C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} 0.$$

   We proceed via induction. To build $s_0 : 0 = C_0 \to C_1$ is trivial; it must be the zero map. We build a nontrivial base case. To build $s_1 : C_1 \to C_2$, note that since free modules are projective, we get s_1 by definition of projective:

$$\begin{tikzcd} & C_1 \arrow[dl, dashed, "s_1"'] \arrow[d, "\mathrm{id}"] & \\ C_2 \arrow[r, "d_2"'] & C_1 \arrow[r, "d_1"'] & 0 \end{tikzcd}$$

   For all subsequent s_n, we use projective-ness again: see that

$$\begin{tikzcd} & C_n = \mathrm{im}(C_{n+1}) \oplus K \arrow[dl, dashed, "s_n"'] \arrow[d, two heads] & \\ C_{n+1} \arrow[r, "d_{n+1}"'] & \mathrm{im}(C_{n+1}) \arrow[r, "d_n"'] & 0 \end{tikzcd}$$

   One can confirm that $d_{n+1} = d_{n+1}s_nd_{n+1}$; see that by projectiveness, the triangle commutes, so

$$\begin{tikzcd} C_{n+1} \arrow[r, "d_{n+1}"] & \mathrm{im}(C_{n+1}) \oplus K \arrow[dl, dashed, "s_n"'] \arrow[d, two heads] & \\ C_{n+1} \arrow[r, "d_{n+1}"'] & \mathrm{im}(C_{n+1}) \arrow[r, "d_n"'] & 0 \end{tikzcd}$$

   and since d_{n+1} is the map $C_{n+1} \to \mathrm{im}(C_{n+1}) \oplus K \twoheadrightarrow \mathrm{im}(C_{n+1})$, we have $d = dsd$, as desired.

2.

Exercise 1.4.2 Let C be a chain complex, with boundaries B_n and cycles Z_n in C_n. Show that C is split if and only if there are R-modules decompositions $C_n \cong Z_n \oplus B'_n$ and $Z_n = B_n \oplus H'_n$. Show that C is split exact iff $H'_n = 0$.

First, assume that C is split; we show the decomposition. We know from exercise 1.3.4 that the following are always short exact sequences:

$$0 \to Z_n \to C_n \xrightarrow{d} B_{n-1} \to 0 \qquad \text{and} \qquad 0 \to B_n \xrightarrow{\partial} Z_n \to Z_n/B_n \to 0.$$

Now as C is split, there exist maps $s_n : C_n \to C_{n+1}$ such that $d = dsd$. Focus on the first short exact sequence. Since $B_{n-1} \subseteq C_{n-1}$, we have a map $s_{n-1}|_{B_{n-1}} : B_{n-1} \to C_n$. As $d_n s_{n-1} d_n = d_n$ by splititude, we see that $d_n\ s_{n-1}|_{B_{n-1}}\ d_n = d_n$. As d_n is surjective onto B_{n-1}, we get $d_n\ s_{n-1}|_{B_{n-1}} = \mathrm{id}_{B_{n-1}}$. Thus, we can invoke the splitting lemma; $C_n \cong Z_n \oplus B_{n-1}$. Let $B'_n = B_{n-1}$ and we are halfway there.

For the other short exact sequence, we're going to use the splitting lemma again once we've constructed a map $Z_n \to B_n$ that composes with ∂ to be id_{B_n}. See that, from the first short exact sequence, we have

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, shift left] & B_n \arrow[r] & 0 \\
0 \arrow[r] & Z_n \arrow[r, "\iota"] & C_n \arrow[r] \arrow[u, shift left, "s_n"] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

Now the map $Z_n \to B_n$ is clear; take $Z_n \overset{\iota}{\hookrightarrow} C_n \xrightarrow{s_n} C_{n+1} \xrightarrow{d_{n+1}} B_n$. Then again, by the splitting lemma, $Z_n \cong B_n \oplus Z_n/B_n$. Let $H'_n = Z_n/B_n$. The result is shown.

•••

Now, assume that we have the given R-module decomposition. We need to show that there exist maps $s_n : C_n \to C_{n+1}$ such that $d = dsd$. Since $C_n \cong Z_n \oplus B'_n \cong B_n \oplus H'_n \oplus B'_n$, if $(b, h', b') \in C_n$, then $d(b, h', b') = (b', 0, 0)$. Define $s : C_n \to C_{n+1}$ to be $s(x, y, z) = (0, 0, x)$. Then we can see that

$$dsd(b, h', b') = ds(b', 0, 0) = d(0, 0, b') = (b', 0, 0) = d(b, h', b'),$$

as desired.

•••

If $H'_n = 0$ and C is split, then $C_n \cong \oplus B_n \oplus B'_n$, and then $\mathrm{im}(d_n) = B_n$, $\ker(d_n) = B'_n = B_{n-1}$. Then obviously $\mathrm{im}(d_n) = B_n = \ker(d_{n+1})$, so C is exact. Conversely, if C is split exact, then $\mathrm{im}(d_n) = B_n = \ker(d_{n+1}) = B_n \oplus H'_n$, so $H'_n = 0$, as desired.

Now suppose that we are given two chain complexes C and D, together with randomly chosen maps $s_n : C_n \to D_{n+1}$. Let f_n be the map from C_n to D_n defined by the formula $f_n = d_{n+1}s_n + s_{n-1}d_n$.

$$\begin{tikzcd}
C_{n+1} \arrow[r, "d"] & C_n \arrow[r, "d"] \arrow[dl, "s"'] \arrow[d, "f"] & C_{n-1} \arrow[dl, "s"'] \\
D_{n+1} \arrow[r, "d"'] & D_n \arrow[r, "d"'] & D_{n-1}
\end{tikzcd}$$

Dropping the subscripts for clarity, we compute

$$df = d(ds + sd) = dsd = (ds + sd)d = fd.$$

Thus $f = ds + sd$ is a chain map from C to D.

Definition 1.4.3 We say that a chain map $f : C \to D$ is *null homotopic* if there are maps $s_n : C_n \to D_{n+1}$ such that $f = ds + sd$. The maps $\{s_n\}$ are called a *chain contraction* of f.

Exercise 1.4.3 Show that C is a split exact chain complex if and only if the identity map on C is null homotopic.

For the first direction, if the identity is null homotopic, then $\mathrm{id} = ds + sd$. Then $d\,\mathrm{id} = d = d(ds + sd) = dds + dsd = dsd$, so C is split. To show exactness, see that since $\mathrm{id} : C \to C$ is null homotopic, the induced map $\mathrm{id}_* : H_n(C) \to H_n(C)$ is the zero map (Lemma 1.4.5). Thus $H_n(C) = 0$, and so C is acyclic.

For the other direction, assume C is split exact. We need to show that there exist $s : C_n \to C_{n+1}$ such that $ds + sd = \mathrm{id}$. As C is split exact, by exercise 1.4.2, $C_n = B_n \oplus B_{n-1} = \mathrm{im}(d_{n+1}) \oplus \mathrm{im}(d_n)$. Then $d : C_n \to C_{n-1}$ is projection onto the $\mathrm{im}(d_n)$ factor and then inclusion into the second coordinate; i.e., $d(x, y) = (0, x)$. Define $s(x, y) = (y, 0)$. Then

$$(ds + sd)(x, y) = ds(x, y) + sd(x, y) = d(y, 0) + s(0, x) = (0, y) + (x, 0) = (x, y) = \mathrm{id}(x, y),$$

so the identity is null homotopic, as desired.

The chain contraction construction gives us an easy way to proliferate chain maps: if $g : C \to D$ is any chain map, so is $g + (sd + ds)$ for *any* choice of maps s_n. However, $g + (sd + ds)$ is not very different from g, in a sense that we shall now explain.

Definition 1.4.4 We say that two chain maps f and g from C to D are *chain homotopic* if their difference $f - g$ is null homotopic, that is, if

$$f - g = sd + ds.$$

The maps $\{s_n\}$ are called a *chain homotopy* from f to g. Finally, we say that $f : C \to D$ is a *chain homotopy equivalence* (Bourbaki uses *homotopism*) if there is a map $g : D \to C$ such that gf and fg are chain homotopic to the respective identity maps of C and D.

Remark This terminology comes from topology via the following observation. A map f between two topological spaces X and Y induces a map $f_* : S(X) \to S(Y)$ between the corresponding singular chain complexes. It turns out that if f is topologically null homotopic (resp. a homotopy equivalence), then the chain map f_* is null homotopic (resp. a chain homotopy equivalence), and if two maps f and g are topologically homotopic, then f_* and g_* are chain homotopic.

Lemma 1.4.5 *If $f : C \to D$ is null homotopic, then every map $f_* : H_n(C) \to H_n(D)$ is zero. If f and g are chain homotopic, then they induce the same maps $H_n(C) \to H_n(D)$.*

Proof. It is enough to prove the first assertion, so suppose that $f = ds + sd$. Every element of $H_n(C)$ is represented by an n-cycle x. But then $f(x) = d(sx)$. That is, $f(x)$ is an n-boundary in D. As such, $f(x)$ represents 0 in $H_n(D)$. $\square$

Exercise 1.4.4 Consider the homology $H_*(C)$ of C as a chain complex with zero differentials. Show that if the complex C is split, then there is a chain homotopy equivalence between C and $H_*(C)$. ~~Give an example in which the converse fails.~~ Conversely, if homotopy equivalent, show that C is split.

Let C be split with splitting maps $s_n : C_n \to C_{n+1}$ where $d = dsd$, and let homology as above; i.e.,

$$\cdots \xrightarrow{0} H_{n+1}(C) \xrightarrow{0} H_n(C) \xrightarrow{0} H_{n-1}(C) \xrightarrow{0} \cdots .$$

We need to show that there is some chain homotopy equivalence $f : C \to H(C)$; i.e., that there exists a $g : H(C) \to C$ such that

$$\mathrm{id}_C - gf = d_C\sigma + \sigma d_C \text{ and}$$
$$\mathrm{id}_{H(C)} - fg = d_{H(C)}\tau + \tau d_{H(C)} = 0\tau + \tau 0 = 0$$

for some $\sigma_n : C_n \to C_{n+1}$ and $\tau_n : H_n(C) \to H_{n+1}(C)$ chain homotopies.

We proceed. First, just let τ be the zero map. As C is split, let $\sigma_n = s_n$. Then, since we know from exercise 1.4.2 that $C_n = B_n \oplus H_n(C) \oplus B_{n-1}$, let $f : C \to H(C)$ take $(x, y, z) \in C_n$ to $y \in H_n(C)$. Then let g map $y \in H_n(C)$ to $(0, y, 0) \in C_n$. Now clearly

$$(\mathrm{id}_{H(C)} - fg)(y) = \mathrm{id}_{H(C)}(y) - fg(y) = y - f(0, y, 0) = y - y = 0 = 0(y)$$

and

$$(\mathrm{id}_C - gf)(x, y, z) = \mathrm{id}_C(x, y, z) - gf(x, y, z) = (x, y, z) - g(y) = (x, y, z) - (0, y, 0) = (x, 0, z),$$

while

$$(ds + sd)(x, y, z) = ds(x, y, z) + sd(x, y, z) = d(0, 0, x) + s(z, 0, 0) = (x, 0, 0) + (0, 0, z) = (x, 0, z),$$

and everything is hunky-dory.

•••

Now assume there exist $f : C \to H(C)$ and $g : H(C) \to C$ such that gf is homotopic to id_C and fg is homotopic to $\mathrm{id}_{H(C)}$. Then as g is a chain map, the square

$$\begin{array}{ccc} H_n(C) & \xrightarrow{0} & H_{n-1}(C) \\ \downarrow g & & \downarrow g \\ C_n & \xrightarrow{d} & C_{n-1} \end{array}$$

commutes and thus $dg = 0$. Now

$$d = d - 0f = d - dgf = d(\mathrm{id}_C - gf) = d(ds + sd) = dds + dsd = dsd,$$

so C is split, as desired.

Exercise 1.4.5 In this exercise we shall show that the chain homotopy classes of maps form a quotient category $\mathbf{K}$ of the category $\mathbf{Ch}$ of all chain complexes. The homology functors H_n on $\mathbf{Ch}$ will factor through the quotient functor $\mathbf{Ch} \to \mathbf{K}$.

1. Show that chain homotopy equivalence is an equivalence relation on the set of all chain maps from C to D. Let $\mathrm{Hom}_{\mathbf{K}}(C, D)$ denote the equivalence classes of such maps. Show that $\mathrm{Hom}_{\mathbf{K}}(C, D)$ is an abelian group.
2. Let f and g be chain homotopic maps from C to D. If $u : B \to C$ and $v : D \to E$ are chain maps, show that vfu and vgu are chain homotopic. Deduce that there is a category $\mathbf{K}$ whose objects are chain complexes and whose morphisms are given in (1).
3. Let f_0, f_1, g_0, and g_1 be chain maps from C to D such that f_i is chain homotopic to g_i $(i = 0, 1)$. Show that $f_0 + f_1$ is chain homotopic to $g_0 + g_1$. Deduce that $\mathbf{K}$ is an additive category, and that $\mathbf{Ch} \to \mathbf{K}$ is an additive functor.
4. Is $\mathbf{K}$ an abelian category? Explain.

1. We show that chain homotopy equivalence is an equivalence relation: reflexive, symmetric, transitive. First, see that $f \sim f$, as $f - f = 0 = d0 + 0d$. Next, if $f \sim g$, then $f - g = ds + sd$, and then $g - f = -(f - g) = -(ds + sd) = -ds - sd = d(-s) + (-s)d$, so $g \sim f$. Finally, if $f \sim g$ and $g \sim h$, then $f - g = ds + sd$ and $g - h = dt + td$. Then $f - h = (f - g) + (g - h) = ds + sd + dt + td = ds + dt + sd + td = d(s + t) + (s + t)d$, so $f \sim h$.

   To see that $\mathrm{Hom}_{\mathbf{K}}(C, D)$ is an abelian group with operation pointwise addition, see that it is associative: $[f]+([g]+[h]) = [f]+[g+h] = [f+g+h] = [f+g]+[h] = ([f]+[g])+[h]$; it has identity $[0]$: $[0] + [f] = [0 + f] = [f]$ and $[f] + [0] = [f + 0] = [f]$ for all f; and it has inverses: $-[f] = [-f]$, because $-[f] + [f] = [-f + f] = [0]$ and $[f] - [f] = [f - f] = [0]$. It is abelian because pointwise addition is: $[f] + [g] = [f + g] = [g + f] = [g] + [f]$.

2. If f and g are chain homotopic, then $f - g = ds + sd$. Then

$$vfu - vgu = v(fu - gu) = v(f-g)u = v(ds+sd)u = v(dsu + sdu) = vdsu + vsdu.$$

As v and u are chain maps, they commute with d, and

$$vdsu + vsdu = dvsu + vsud = d(vsu) + (vsu)d,$$

so vfu is chain homotopic to vgu.

To check $\mathbf{K}$ is a category, we need to show composition is associative and there is an identity for each chain complex. The first is easy; since composition of equivalence classes is equivalence classes of composition by above, composition is associative. For the second, take $[\mathrm{id}_{C_\bullet}]$ to be the identity. Then if $[f] : B_\bullet \to C_\bullet$ or $[g] : C_\bullet \to D_\bullet$, then $[\mathrm{id}][f] = [\mathrm{id}\, f] = [f]$ and $[g][\mathrm{id}] = [g\,\mathrm{id}] = [g]$.

3. If f_i is chain homotopic to g_i, then $f_i - g_i = ds_i + s_i d$. Then

$$\begin{aligned} f_0 + f_1 - g_0 + g_1 = (f_0 - g_0) + (f_1 - g_1) &= ds_0 + s_0 d + ds_1 + s_1 d \\ &= ds_0 + ds_1 + s_0 d + s_1 d \\ &= d(s_0 + s_1) + (s_0 + s_1)d, \end{aligned}$$

and $f_0 + f_1$ is chain homotopic to $g_0 + g_1$.

K is an additive category because

- **K** has zero object the zero complex,
- **K** has products $C_\bullet \times D_\bullet$;

and $F : \mathbf{Ch} \to \mathbf{K}$ is an additive functor because

- it is a functor:
  * it takes identity maps in **Ch** to equivalence classes of identity maps, which are identity maps in **K**, and
  * it respects composition by 2.: $F(f) \circ F(g) = [f] \circ [g] = [f \circ g] = F(f \circ g)$;
- and it is additive: $\mathrm{Hom}_{\mathbf{Ch}}(C, C') \to \mathrm{Hom}_{\mathbf{K}}(FC, FC')$ is a group homomorphism. Indeed, $f \mapsto [f]$ is a homomorphism, because $[f+g] = [f] + [g]$.

4. I'm told no; one should check that one of the following fails:

(a) every map in $\mathbf{K}$ has a kernel and cokernel,
(b) every monic in $\mathbf{K}$ is the kernel of its cokernel,
(c) every epi in $\mathbf{K}$ is the cokernel of its kernel.

1.5 Mapping Cones and Cylinders

1.5.1 Let $f : B \to C$ be a map of chain complexes. The *mapping cone* of f is the chain complex $\mathrm{cone}(f)$ whose degree n part is $B_{n-1} \oplus C_n$. In order to match other sign conventions, the differential in $\mathrm{cone}(f)$ is given by the formula

$$d(b,c) = (-d(b), d(c) - f(b)), \qquad\qquad (b \in B_{n-1}, c \in C_n).$$

That is, the differential is given by the matrix

$$\begin{bmatrix} -d_B & 0 \\ -f & +d_C \end{bmatrix} : \qquad \begin{array}{ccc} B_{n-1} & \xrightarrow{-} & B_{n-2} \\ \oplus & \searrow^{-} & \oplus \\ C_n & \xrightarrow{+} & C_{n-1} \end{array}$$

Here is the dual notion for a map $f : B \to C$ of cochain complexes. The mapping cone, $\mathrm{cone}(f)$, is a cochain complex whose degree n part is $B^{n+1} \oplus C^n$. The differential is given by the same formula as above with the same signs.

Exercise 1.5.1 Let $\mathrm{cone}(C)$ denote the mapping cone of the identity map id_C of C; it has $C_{n-1} \oplus C_n$ in degree n. Show that $\mathrm{cone}(C)$ is split exact, with $s(b,c) = (-c, 0)$ defining the splitting map.

Explicitly,

$$\mathrm{cone}(C) : \qquad \cdots \to C_n \oplus C_{n+1} \to C_{n-1} \oplus C_n \to C_{n-2} \oplus C_{n-1} \to \cdots$$

with differential

$$d(b,c) = (-d_C(b), d_C(c) - \mathrm{id}(b)) = (-db, dc - b).$$

To see $\mathrm{cone}(C)$ is exact, see that

$$dd(b,c) = d(-db, dc-b) = \Big(- d\big(- db\big), d\big(dc - b\big) - \big(- db\big)\Big) = (ddb, ddc - db + db) = (0,0).$$

To see cone(C) is split, we use the map s given $(s(b,c)=(-c,0))$. Then observe that

$$\begin{aligned} dsd(b,c) = ds(-db, dc-b) = d(b-dc, 0) &= (-d(b-dc), d0-(b-dc)) \\ &= (-db+ddc, 0-b+dc) \\ &= (-db, dc-b) \\ &= d(b,c). \end{aligned}$$

Exercise 1.5.2 Let $f : C \to D$ be a map of complexes. Show that f is null homotopic if and only if f extends to a map $(-s,f) : \text{cone}(C) \to D$.

When we say "f extends to a map $(-s,f) : \text{cone}(C) \to D$," we mean that such an $(-s,f)$ is a chain map.

Suppose f is null homotopic. Then there exist $s_n : C_n \to D_{n+1}$ such that $f = ds+sd$. Let s in the extension be s the chain contraction. Then see that $(-s,f) : \text{cone}(C)_n = C_{n-1} \oplus C_n \to D_{n+1}$ takes (x,y) to $-s(x)+f(y)$. To see that $(-s,f)$ is a chain map, see that

$$d(-s,f)(x,y) = d(-sx+fy) = -dsx+dfy = sdx - fx + dfy$$

and

$$(-s,f)d(x,y) = (-s,f)(-dx, dy-x) = -s(-dx) + f(dy-x) = sdx + fdy - fx.$$

Since f is a chain map, f commutes with d, so

$$d(-s,f)(x,y) = sdx - fx + dfy = sdx - fx + fdy = (-s,f)d(x,y),$$

and $(-s,f)$ is a chain map, as desired.

Now suppose that we have a chain map $(t,f) : \text{cone}(C) \to D$. We need to show that f is null homotopic. Indeed, such chain contractions will be $-t$. See that since $d(t,f) = (t,f)d$, we have

$$-tdx + fdy - fx = (t,f)(-dx, dy-x) = (t,f)d(x,y) = d(t,f)(x,y) = d(tx+fy) = dtx + dfy.$$

So $-tdx + fdy - fx = dtx + dfy$. As f is a chain map, $dfy = fdy$, so

$$-tdx + fdy - fx = dtx + dfy$$

$$-tdx - fx = dtx$$

$$-dtx - tdx = fx$$

$$d(-t)x + (-t)dx = fx,$$

and f is null homotopic with chain contraction $-t$, as desired.

1.5.2 Any map $f_* : H_*(B) \to H_*(C)$ can be fit into a long exact sequence of homology groups by use of the following device. There is a short exact sequence

$$0 \to C \to \operatorname{cone}(f) \xrightarrow{\delta} B[-1] \to 0$$

of chain complexes, where the left map sends c to $(0, c)$, and the right map sends (b, c) to $-b$. Recalling (1.2.8) that $H_{n+1}(B[-1]) \cong H_n(B)$, the homology long exact sequence (with connecting homomorphism ∂) becomes

$$\cdots \to H_{n+1}(\operatorname{cone}(f)) \xrightarrow{\delta_*} H_n(B) \xrightarrow{\partial} H_n(C) \to H_n(\operatorname{cone}(f)) \xrightarrow{\delta_*} H_{n-1}(B) \xrightarrow{\partial} \cdots.$$

The following lemma shows that $\partial = f_*$, fitting f_* into a long exact sequence.

Lemma 1.5.3 *The map ∂ in the above sequence is f_*.*

Proof. If $b \in B_n$ is a cycle, the element $(-b, 0)$ in the cone complex lifts b via δ. Applying the differential we get $(db, fb) = (0, fb)$. This shows that

$$\partial[b] = [fb] = f_*[b].$$

□

Corollary 1.5.4 *A map $f : B \to C$ is a quasi-isomorphism if and only if the mapping cone complex* cone(f) *is exact. This device reduces questions about quasi-isomorphisms to the study of exact complexes.*

Topological Remark Let K be a simplicial complex (or more generally a cell complex). The *topological cone* CK of K is obtained by adding a new vertex s to K and "coning off" the simplices (cells) to get a new $(n+1)$-simplex for every old n-simplex of K. (See Figure 1.1.) The simplicial (cellular) chain complex $C_\bullet(s)$ of the one-point space $\{s\}$ is R in degree 0 and zero elsewhere. $C_\bullet(s)$ is a subcomplex of the simplicial (cellular) chain complex $C_\bullet(CK)$ of the topological cone CK. The quotient $C_\bullet(CK)/C_\bullet(s)$ is the chain complex $\operatorname{cone}(C_\bullet K)$ of the identity map of $C_\bullet(K)$. The algebraic fact that $\operatorname{cone}(C_\bullet K)$ is split exact (null homotopic) reflects the fact that the topological cone CK is contractible.

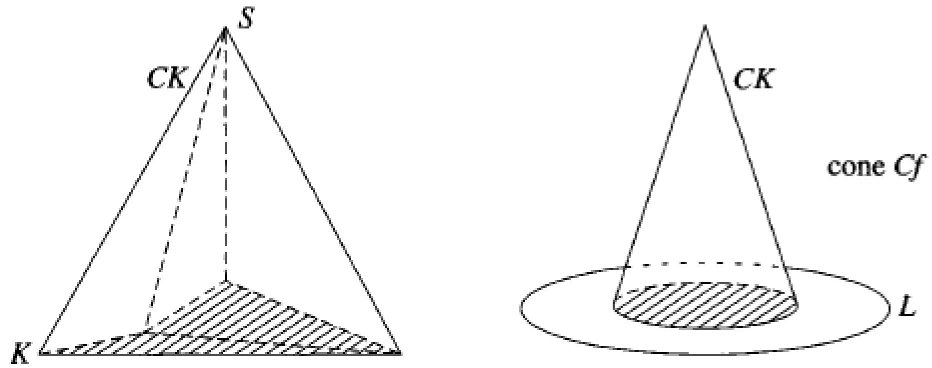


Figure 1.1. The topological cone CK and mapping cone Cf.

More generally, if $f : K \to L$ is simplicial map (or a cellular map), the *topological mapping cone* Cf of f is obtained by glueing CK and L together, identifying the subcomplex K of CK with its image in L (Figure 1.1). This is a cellular complex, which is simplicial if f is an inclusion of simplicial complexes. Write $C_\bullet(Cf)$ for the cellular chain complex of the topological mapping cone Cf. The quotient chain complex $C_\bullet(Cf)/C_\bullet(s)$ may be identified with cone(f_*), the mapping cone of the chain map $f_* : C_\bullet(K) \to C_\bullet(L)$.

1.5.5 A related construction is that of the *mapping cylinder* $\mathrm{cyl}(f)$ of a chain complex map $f : B_\bullet \to C_\bullet$. The degree n part of $\mathrm{cyl}(f)$ is $B_n \oplus B_{n-1} \oplus C_n$, and the differential is

$$d(b, b', c) = (d(b) + b', -d(b'), d(c) - f(b')).$$

That is, the differential is given by the matrix

$$\begin{bmatrix} d_B & \mathrm{id}_B & 0 \\ 0 & -d_B & 0 \\ 0 & -f & d_C \end{bmatrix} : \qquad \begin{tikzcd} B_n \arrow[r, "+"] & B_{n-1} \\ \oplus & \oplus \\ B_{n-1} \arrow[r, "-"] \arrow[uur, "+"] \arrow[ddr, "-"] & B_{n-2} \\ \oplus & \oplus \\ C_n \arrow[r, "+"] & C_{n-1} \end{tikzcd}$$

The cylinder is a chain complex because

$$d^2 = \begin{bmatrix} {d_B}^2 & d_B - d_B & 0 \\ 0 & {d_B}^2 & 0 \\ 0 & f d_B - d_C f & {d_C}^2 \end{bmatrix} = 0.$$

Exercise 1.5.3 Let $\mathrm{cyl}(C)$ denote the mapping cylinder of the identity map id_C of C; it has $C_n \oplus C_{n-1} \oplus C_n$ in degree n. Show that two chain maps $f, g : C \to D$ are chain homotopic if and only if they extend to a map $(f, s, g) : \mathrm{cyl}(C) \to D$.
If $f : B \to C$, $g : C \to D$ and $e : B \to D$ are chain maps, show that e and gf are chain homotopic if and only if there is a chain map $\gamma = (e, s, g)$ from $\mathrm{cyl}(f)$ to D. Note that e and g factor through γ.

Again, extending means the extension is a chain map.

First, suppose f is chain homotopic to g, so $f - g = ds + sd$. Then let the s of the chain homotopy be the s of the extension. We show $d(f,s,g) = (f,s,g)d$. See that

$$d(f,s,g)(x,y,z) = d(fx + sy + gz) = dfx + dsy + dgz, \text{ and}$$
$$(f,s,g)d(x,y,z) = (f,s,g)(dx + y, -dy, dz - \mathrm{id}\, y) = fdx + fy - sdy + gdz - gy.$$

Using the chain homotopy, $fy = gy + dsy + sdy$, so

$$\begin{aligned} fdx + fy - sdy + gdz - gy &= fdx + gy + dsy + sdy - sdy + gdz - gy \\ &= fdx + dsy + gdz. \end{aligned}$$

As f and g are chain maps, they commute with d:

$$fdx + dsy + gdz = dfx + dsy + dgz,$$

and $d(f,s,g) = (f,s,g)d$, as desired.

In the other direction, assume that $(f,t,g) : \mathrm{cyl}(C) \to D$ is a chain map. We show that f is chain homotopic to g, and indeed the chain homotopy is the same s. To see this, observe that since $d(f,s,g) = (f,s,g)d$, we have

$$\begin{aligned} &dfx + dsy + dgz = \\ &d(fx + sy + gz) = \\ &d(f,s,g)(x,y,z) = (f,s,g)d(x,y,z) \\ &\qquad = (f,s,g)(dx + y, -dy, dz - y) \\ &\qquad = fdx + fy - sdy + gdz - gy, \end{aligned}$$

so $dfx + dsy + dgz = fdx + fy - sdy + gdz - gy$. As f and g are chain maps, we commute them with d, so

$$\begin{aligned}
dfx + dsy + dgz &= fdx + fy - sdy + gdz - gy \\
dsy &= fy - sdy - gy \\
dsy + sdy &= fy - gy,
\end{aligned}$$

and f is chain homotopic to g with chain homotopy s, as desired.

• • •

For the second question, first assume that e and gf are chain homotopic; then $e - gf = ds + sd$. Then we need to show $d(e,s,g) = (e,s,g)d$ between $\mathrm{cyl}(f)$ and D. See that

$$\begin{aligned}
&d(e,s,g)(x,y,z) = d(ex+sy+gz) = dex + dsy + dgz, \text{ and} \\
&(e,s,g)d(x,y,z) = (e,s,g)(dx+y,-dy,dz-fy) = edx + ey - sdy + gdz - gfy.
\end{aligned}$$

Commute the chain maps with the differentials and use $ey - gfy = dsy + sdy$:

$$edx + ey - sdy + gdz - gfy = dex + dsy + sdy - sdy + dgz = dex + dsy + dgz,$$

so $d(e,s,g) = (e,s,g)d$.

In the other direction, suppose $(e,s,g) : \mathrm{cyl}(f) \to D$ is a chain map. We show $e - gf = ds + sd$ for the same s. See that by virtue of being a chain map,

$$\begin{aligned}
dex + dsy + dgz &= \\
d(ex + sy + gz) &= \\
d(e,s,g)(x,y,z) &= (e,s,g)d(x,y,z) \\
&= (e,s,g)(dx+y,-dy,dz-fy) \\
&= edx + ey - sdy + gdz - gfy.
\end{aligned}$$

Commute chain maps:

$$dex + dsy + dgz = edx + ey - sdy + gdz - gfy$$
$$dsy = ey - sdy - gfy$$
$$dsy + sdy = ey - gfy.$$

Boom done.

Lemma 1.5.6 *The subcomplex of elements* $(0,0,c)$ *is isomorphic to* C*, and the corresponding inclusion* $\alpha : C \to \mathrm{cyl}(f)$ *is a quasi-isomorphism.*

Proof. The quotient ${}^{\mathrm{cyl}(f)}\!/_{\alpha(C)}$ is the mapping cone of $-\mathrm{id}_B$, so it is null homotopic (exercise 1.5.1). The lemma now follows from the long exact homology sequence for

$$0 \to C \xrightarrow{\alpha} \mathrm{cyl}(f) \to \mathrm{cone}(-\mathrm{id}_B) \to 0.$$

□

Exercise 1.5.4 Show that $\beta(b, b', c) = f(b) + c$ defines a chain map from $\mathrm{cyl}(f)$ to C such that $\beta\alpha = \mathrm{id}_C$. Then show that the formula $s(b, b', c) = (0, b, 0)$ defines a chain homotopy from the identity of $\mathrm{cyl}(f)$ to $\alpha\beta$. Conclude that α is in fact a chain homotopy equivalence between C and $\mathrm{cyl}(f)$.

First, β is a chain map: see that

$$d\beta(x, y, z) = d(fx + z) = dfx + dz;$$
$$\beta d(x, y, z) = \beta(dx + y, -dy, dz - fy) = fdx + fy + dz - fy = fdx + dz.$$

f is a chain map and thus commutes with d, so β is a chain map.

Next, see that $\beta\alpha = \mathrm{id}_C$, since:

$$\beta\alpha(x) = \beta(0, 0, x) = f(0) + x = x = \mathrm{id}_C(x).$$

Now, we need to show that the given s is a chain homotopy from $\mathrm{id}_{\mathrm{cyl}(f)}$ to $\alpha\beta$. See that

$$\mathrm{id}(x, y, z) - \alpha\beta(x, y, z) = (x, y, z) - \alpha(fx + z) = (x, y, z) - (0, 0, fx + z) = (x, y, -fx),$$

and

$$\begin{aligned} ds(x,y,z)+sd(x,y,z) &= d(0,x,0)+s(dx+y,-dy,dz-fy) \\ &= (d0+x,-dx,d0-fx)+(0,dx+y,0) \\ &= (x,y,-fx). \end{aligned}$$

Now, we can conclude that $\alpha : C \to \text{cyl}(f)$ is a chain homotopy equivalence, because the map $\beta : \text{cyl}(f) \to C$ is such that $\alpha\beta$ and $\beta\alpha$ are chain homotopic/equal to (hence chain homotopic) to $\text{id}_{\text{cyl}(f)}$ and id_C, respectively, by above.

Topological Remark Let X be a cellular complex and let I denote the interval $[0,1]$. The space $I \times X$ is the topological cylinder of X. It is also a cell complex; every n-cell e^n in X gives rise to three cells in $I \times X$: the two n-cells, $0 \times e^n$ and $1 \times e^n$, and the $(n+1)$-cell $(0,1) \times e^n$. If $C_\bullet(X)$ is the cellular chain complex of X, then the cellular chain complex $C_\bullet(I \times X)$ of $I \times X$ may be identified with $\text{cyl}(\text{id}_{C_\bullet X})$, the mapping cylinder chain complex of the identity map on $C_\bullet(X)$.

More generally, if $f : X \to Y$ is a cellular map, then the topological mapping cylinder $\text{cyl}(f)$ is obtained by glueing $I \times X$ and Y together, identifying $0 \times X$ with the image of X under f (see Figure 1.2). This is also a cellular complex, whose cellular chain complex $C_\bullet(\text{cyl}(f))$ may be identified with the mapping cylinder of the chain map $C_\bullet(X) \to C_\bullet(Y)$.

The constructions in theis section are the algebraic analogues of the usual topological constructions $I \times X \simeq X$, $\text{cyl}(f) \simeq Y$, and so forth which were used by Dold and Puppe to get long exact sequences for any generalized homology theory on topological spaces.

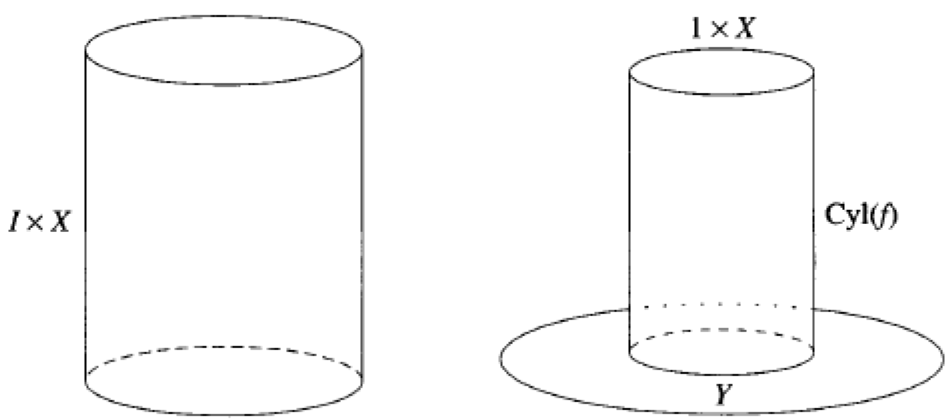


Figure 1.2. The topological cylinder of X and mapping cylinder $\text{cyl}(f)$.

Here is how to use mapping cylinders to fit f_* into a long exact sequence of homology groups. The subcomplex of elements $(b,0,0)$ in $\text{cyl}(f)$ is isomorphic to B, and the quotient ${\text{cyl}(f)}/{B}$ is the mapping cone of f. The composite $B \to \text{cyl}(f) \xrightarrow{\beta} C$ is the map f, where β is the equivalence of exercise 1.5.4, so on homology $f_* : H(B) \to H(C)$ factors through $H(B) \to H(\text{cyl}(f))$. Therefore we may construct a commutative diagram of chain complexes with exact rows:

$$\begin{tikzcd}
& & C & & & \\
0 \arrow[r] & B \arrow[r] \arrow[ur, "f"] & \mathrm{cyl}(f) \arrow[r] \arrow[u, "\beta"] & \mathrm{cone}(f) \arrow[r] \arrow[d, equal] & 0 & \\
& 0 \arrow[r] & C \arrow[r] \arrow[u, "\alpha"] & \mathrm{cone}(f) \arrow[r, "\delta"] & B[-1] \arrow[r] & 0.
\end{tikzcd}$$

The homology long exact sequences fit into the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "-\partial"] & H_n(B) \arrow[r] \arrow[d, equal] \arrow[dr, "f"] & H_n(\mathrm{cyl}(f)) \arrow[r] \arrow[d, equal, "\sim"] & H_n(\mathrm{cone}(f)) \arrow[r, "-\partial"] \arrow[d, equal] & H_{n-1}(B) \arrow[r] \arrow[d, equal, "\sim"] & \cdots \\
\cdots \arrow[r] & H_{n+1}(B[-1]) \arrow[r, "\partial"'] & H_n(C) \arrow[r] & H_n(\mathrm{cone}(f)) \arrow[r, "\delta"] & H_n(B[-1]) \arrow[r, "\partial"'] & \cdots
\end{tikzcd}$$

Lemma 1.5.7 *This diagram is commutative, with exact rows.*

Proof. It suffices to show that the right square (with $-\partial$ and δ) commutes. Let (b, c) be an n-cycle in $\mathrm{cone}(f)$, so $d(b) = 0$ and $f(b) = d(c)$. Lift it to $(0, b, c)$ in $\mathrm{cyl}(f)$ and apply the differential:

$$d(0, b, c) = (0 + b, -db, dc - fb) = (b, 0, 0).$$

Therefore ∂ maps the class of (b, c) to the class of $b = -\delta(b, c)$ in $H_{n-1}(B)$. $\square$

1.5.8 The cone and cylinder constructions provide a natural way to fit the homology of *every* chain map $f : B \to C$ into *some* long exact sequence (see 1.5.2 and 1.5.7). To show that the long exact sequence is well defined, we need to show that the usual long exact homology sequence attached to any short exact sequence of complexes

$$0 \to B \xrightarrow{f} C \xrightarrow{g} D \to 0$$

agrees both with the long exact sequence attached to f and with the long exact sequence attached to g.

We first consider the map f. There is a chain map $\varphi : \mathrm{cone}(f) \to D$ defined by the formula $\varphi(b, c) = g(c)$. It fits into a commutative diagram with exact rows:

$$\begin{tikzcd}
& 0 \arrow[r] & C \arrow[r] \arrow[d, "\alpha"] & \mathrm{cone}(f) \arrow[r, "\delta"] \arrow[d, equal] & B[-1] \arrow[r] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & \mathrm{cyl}(f) \arrow[r] \arrow[d, "\beta"] & \mathrm{cone}(f) \arrow[r] \arrow[d, "\varphi"] & 0 & \\
0 \arrow[r] & B \arrow[r, "f"] & C \arrow[r, "g"] & D \arrow[r] & 0. &
\end{tikzcd}$$

Since β is a quasi-isomorphism, it follows from the 5-lemma and 1.3.4 that φ is a quasi-isomorphism as well. The following exercise shows that φ need not be a chain homotopy equivalence.

Exercise 1.5.5 Suppose that the B and C of 1.5.8 are modules, considered as chain complexes concentrated in degree zero. Then $\mathrm{cone}(f)$ is the complex $0 \to B \xrightarrow{-f} C \to 0$. Show that φ is a chain homotopy equivalence iff $f : B \subseteq C$ is a split injection.

Above, we defined $\varphi : \mathrm{cone}(f) \to D$ to be $\varphi(b, c) = g(c)$. First, suppose φ is a chain homotopy equivalence. Then there exists some map $\psi : D \to \mathrm{cone}(f)$ such that $\varphi\psi$ is chain homotopic to id_D and $\psi\varphi$ is chain homotopic to $\mathrm{id}_{\mathrm{cone}(f)}$; i.e., $\mathrm{id}_D - \varphi\psi = ds + sd$ and $\mathrm{id}_{\mathrm{cone}(f)} - \psi\varphi = dt + td$.

We need to show that $f : B \hookrightarrow C$ is split; that is, there exists a map $\widetilde{f} : C \to B$ such that $\widetilde{f} f = \mathrm{id}_B$ (see "Construction of Ext").

First, notice that in the following diagram, the down maps are the chain map φ, which is only nontrivial in the $C \xrightarrow{g} D_0$ column.

$$\begin{tikzcd} & 0 \arrow[r] & B \arrow[r, "-f"] \arrow[d, "\varphi=0"] & C \arrow[r] \arrow[d, "\varphi=g"] & 0 \\ {} \arrow[r] & D_2 \arrow[r] & D_1 \arrow[r] & D_0 \arrow[r] & D_{-1} \arrow[r] & {} \end{tikzcd}$$

By chain homotopy equivalence, $\mathrm{id}_{\mathrm{cone}(f)} - \psi\varphi = dt + td$, which means

$$\begin{tikzcd} & B \arrow[r, "-f"] \arrow[dl, "t"'] \arrow[d, "\mathrm{id}_B - \psi 0"] & C \arrow[dl, "t"] \\ 0 \arrow[r, "0"'] & B \end{tikzcd}$$

This means given $b \in B$, we have

$$b = b - \psi 0(b) = (\mathrm{id}_{\mathrm{cone}(f)} - \psi\varphi)(b) = (dt + td)(b) = 0t(b) + t(-f)(b) = -tf(b),$$

or in other words, $\mathrm{id}_B = -tf$. So let $\widetilde{f} = -t : C \to B$, and thus f is a split injection.

• • •

On the other hand, suppose f is a split injection. Then there exists $\widetilde{f} : C \to B$ such that $\widetilde{f} f = \mathrm{id}_B$. We need to show that φ is a chain homotopy equivalence, so we need to produce a $\psi : D \to \mathrm{cone}(f)$. By "Construction of Ext," f is split if and only if g is split, so there exists $\widetilde{g} : D \to C$ such that $g\widetilde{g} = \mathrm{id}_D$. Let ψ be the vertical maps (noting that the only nontrivial one is $\psi : D_0 \to C$)

$$\begin{tikzcd} & 0 \arrow[r] & B \arrow[r, "-f"] & C \arrow[r] & 0 \\ {} \arrow[r] & D_2 \arrow[r] & D_1 \arrow[r] \arrow[u, "\psi=0"] & D_0 \arrow[r] \arrow[u, "\psi=\widetilde{g}|_{D_0}"] & D_{-1} \arrow[r] & {} \end{tikzcd}$$

Then see that to confirm $\mathrm{id}_{\mathrm{cone}(f)} - \psi\varphi = dt + td$, we have

$$\begin{tikzcd} 0 \arrow[r] \arrow[dd, "0"] & B \arrow[rr, "-f"] \arrow[dr, "0"] \arrow[dd, "\mathrm{id}_B"] \arrow[ddl, "t"'] & & C \arrow[rr] \arrow[dr, "g"] \arrow[dd, "\mathrm{id}_C"] \arrow[ddl, "t"'] & & 0 \arrow[dd, "0"] \arrow[ddl, "t"'] \\ & & D_1 \arrow[dl, "0"] & & D_0 \arrow[dl, "\widetilde{g}"] & \\ 0 \arrow[r] & B \arrow[rr, "-f"'] & & C \arrow[rr] & & 0 \end{tikzcd}$$

The only nontrivial chain contraction t is $t : C \to B$, which we'll declare to be $t = -\widetilde{f}$. Then see that

$$\begin{aligned} \mathrm{id}_B - \psi\varphi &= \mathrm{id}_B - 00 = \mathrm{id}_B = \widetilde{f}f = 00 + (-\widetilde{f})(-f) = dt + td, \text{ and} \\ \mathrm{id}_C - \psi\varphi &= \mathrm{id}_C - \widetilde{g}g = ... = f\widetilde{f} = (-f)(-\widetilde{f}) + 00 = dt + td. \end{aligned}$$

If we can show that there is a way to connect the "..." in the above, then we have it. Indeed, given $x \in C$, we claim $(\mathrm{id}_C - \widetilde{g}g)(x) = f\widetilde{f}(x)$. To see this, observe that

$$g(\mathrm{id}_C - \widetilde{g}g)(x) = g(x - \widetilde{g}g(x)) = g(x) - g\widetilde{g}g(x) = g(x) - g(x) = 0,$$

so $(\mathrm{id}_C - \widetilde{g}g)(x) \in \ker g = \operatorname{im} f$. Thus there exists $y \in B$ such that $f(y) = (\mathrm{id}_C - \widetilde{g}g)(x)$. We claim that $y = \widetilde{f}(\mathrm{id}_C - \widetilde{g}g)(x)$. To see this, note that

$$\begin{aligned} f(y) &= (\mathrm{id}_C - \widetilde{g}g)(x) \\ \widetilde{f}f(y) &= \widetilde{f}(\mathrm{id}_C - \widetilde{g}g)(x) \\ y &= \widetilde{f}(\mathrm{id}_C - \widetilde{g}g)(x), \end{aligned}$$

as claimed. Then

$$\begin{aligned} (\mathrm{id}_C - \widetilde{g}g)(x) &= f\widetilde{f}(\mathrm{id}_C - \widetilde{g}g)(x) \\ (\mathrm{id}_C - \widetilde{g}g)(x) &= f\widetilde{f}(x) - f\widetilde{f}\widetilde{g}g(x), \end{aligned}$$

so if $f\widetilde{f}\widetilde{g}g = 0$, then we're done. To see this, we claim that $\widetilde{f}\widetilde{g} = 0$.

> To prove the claim, see that as $0 \to B \xrightarrow{f} C \xrightarrow{g} D \to 0$ is split exact, we have by "Construction of Ext" the commutative diagram
>
> $$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, "f"'] \arrow[d, "\mathrm{id}_B"'] & C \arrow[l, bend right, "\widetilde{f}"'] \arrow[r, "g"'] \arrow[d, dashed, "\theta"] & D \arrow[l, bend right, "\widetilde{g}"'] \arrow[r] \arrow[d, "\mathrm{id}_D"] & 0 \\ 0 \arrow[r] & B \arrow[r, "\iota_1"] & B \oplus D \arrow[l, bend left, "\pi_1"] \arrow[r, "\pi_2"] & D \arrow[l, bend left, "\iota_2"] \arrow[r] & 0 \end{tikzcd}$$

Now by the diagram, for all $x \in D$,

$$\widetilde{f}\widetilde{g}(x) = {\mathrm{id}_B}^{-1}\pi_1\iota_2 \,\mathrm{id}_D(x) = \pi_1\iota_2(x) = \pi_1(0, x) = 0,$$

and the result is shown.

Thus, $f\widetilde{f}\widetilde{g}g = f0g = 0$ as we needed to show. Therefore, $\mathrm{id}_{\mathrm{cone}(f)} - \psi\varphi = dt + td$.

Now, we need to show $\mathrm{id}_D - \varphi\psi = ds + sd$. Here, we note something critical that we omitted in the above steps, as it wasn't necessary for the proof and thus we've shown things above in a bit more generality. The critical thing to note is that since $0 \to B \to C \to D \to 0$ is exact, it is by exercise 1.2.4 that $0 \to B_n \to C_n \to D_n \to 0$ is exact for all n. Since B_n and C_n are only nonzero in degree 0, that forces D_n to be trivial in all but degree 0 as well.

So we have:

$$\begin{tikzcd}
0 \arrow[r] \arrow[dd, bend right, "\mathrm{id}_D=0"'] \arrow[d, "0"] & D_0 \arrow[r] \arrow[ddl, "s"'] \arrow[dd, bend right, "\mathrm{id}_D"'] \arrow[d, "\widetilde{g}"] & 0 \arrow[ddl, "s"'] \arrow[dd, "\mathrm{id}_D=0"] \\
B \arrow[d, "0"] & C \arrow[d, "g"] & \\
0 \arrow[r] & D_0 \arrow[r] & 0
\end{tikzcd}$$

Now, if we let the chain contractions be $\{s = 0 : D_n \to D_{n+1}\}$ for all n, then we get

$$\mathrm{id}_{D_k} - \varphi\psi = 0 - 00 = 0 = 00 + 00 = ds + sd \text{ for all } k \neq 0, \text{ and}$$
$$\mathrm{id}_{D_0} - \varphi\psi = \mathrm{id}_{D_0} - g\widetilde{g} = \mathrm{id}_{D_0} - \mathrm{id}_{D_0} = 0 = 00 + 00 = ds + sd.$$

Thus f split implies φ is a chain homotopy equivalence, as desired.

To continue, the naturality of the connecting homomorphism ∂ provides us with a natural isomorphism of long exact sequences:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & H_n(B) \arrow[r] \arrow[d, equal] & H_n(\mathrm{cyl}(f)) \arrow[r] \arrow[d, "\cong"] & H_n(\mathrm{cone}(f)) \arrow[r, "\partial"] \arrow[d, "\cong"] & H_{n-1}(B) \arrow[r] \arrow[d, equal, "\sim"] & \cdots \\
\cdots \arrow[r, "\partial"] & H_n(B) \arrow[r] & H_n(C) \arrow[r] & H_n(D) \arrow[r, "\partial"] & H_{n-1}(B) \arrow[r] & \cdots .
\end{tikzcd}$$

Exercise 1.5.6 Show that the composite

$$H_n(D) \cong H_n(\mathrm{cone}(f)) \xrightarrow{-\delta_*} H_n(B[-1]) \cong H_{n-1}(B)$$

is the connecting homomorphism ∂ in the homology long exact sequence for

$$0 \to B \to C \to D \to 0.$$

By Proposition 1.3.4, given an arbitrary commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

we get a commutative diagram with exact rows

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & H_n(A) \arrow[r] \arrow[d] & H_n(B) \arrow[r] \arrow[d] & H_n(C) \arrow[r, "\partial"] \arrow[d] & H_{n-1}(A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r, "\partial'"] & H_n(A') \arrow[r] & H_n(B') \arrow[r] & H_n(C') \arrow[r, "\partial'"] & H_{n-1}(A') \arrow[r] & \cdots
\end{tikzcd}$$

In this case, see that we have

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & \operatorname{cyl}(f) \arrow[r] \arrow[d, "\beta"] & \operatorname{cone}(f) \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & B \arrow[r, "f"] & C \arrow[r, "g"] & D \arrow[r] & 0.
\end{tikzcd}$$

so by 1.3.4 we get

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(B) \arrow[r] \arrow[d, equal] & H_n(\operatorname{cyl}(f)) \arrow[r] \arrow[d, "\wr"] & H_n(\operatorname{cone}(f)) \arrow[r] \arrow[d, "\wr"] & H_{n-1}(B) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r, "\partial"] & H_n(B) \arrow[r] & H_n(C) \arrow[r] & H_n(D) \arrow[r, "\partial"] & H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

but from lemma 1.5.7 we also have the commutative diagram with exact rows

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(B[-1]) \arrow[r] \arrow[d, equal] & H_n(C) \arrow[r] \arrow[d, equal, "\wr"] & H_n(\operatorname{cone}(f)) \arrow[r, "-\delta_*"] \arrow[d, equal] & H_n(B[-1]) \arrow[r] \arrow[d, equal, "\wr"] & \cdots \\
\cdots \arrow[r] & H_n(B) \arrow[r] \arrow[ur, "f"] & H_n(\operatorname{cyl}(f)) \arrow[r] & H_n(\operatorname{cone}(f)) \arrow[r] & H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

Gluing together these two commutative diagrams, we get

$$\begin{tikzcd}
{} \arrow[r] & H_n(\operatorname{cone}(f)) \arrow[r, "-\delta_*"] \arrow[d, equal] & H_n(B[-1]) \arrow[r] \arrow[d, equal] & {} \\
{} \arrow[r] & H_n(\operatorname{cone}(f)) \arrow[r] \arrow[d, "\wr"] & H_{n-1}(B) \arrow[r] \arrow[d, equal] & {} \\
{} \arrow[r] & H_n(D) \arrow[r, "\partial"] & H_{n-1}(B) \arrow[r] & {}
\end{tikzcd}$$

And since the diagram commutes,

$$\partial : H_n(D) \to H_{n-1}(B) = H_n(D) \cong H_n(\operatorname{cone}(f)) \xrightarrow{-\delta_*} H_n(B[-1]) \cong H_{n-1}(B),$$

Exercise 1.5.7 Show that there is a quasi-isomorphism $B[-1] \to \operatorname{cone}(g)$ dual to φ. Then dualize the preceding exercise, by showing that the composite

$$H_n(D) \xrightarrow{\partial} H_{n-1}(B) \xrightarrow{\simeq} H_n(\text{cone}(g))$$

is the usual map induced by the inclusion of D in cone(g).

First, we show that there is a quasi-isomorphism $\psi : B[-1] \to \text{cone}(g)$ dual to φ. Recall since $g : C \to D$, $\text{cone}(g)_n = C_{n-1} \oplus D_n$. Define ψ at degree n by $\psi(b_{n-1}) = (-f(b_{n-1}), 0)$.

Replacing all $f : B \to C$ with $g : C \to D$ in a previous diagram, we have

$$\begin{tikzcd}
 & & 0 \arrow[r] & B[-1] \arrow[r, "f"] \arrow[d, "\psi"] \arrow[dr, "f"] & C[-1] \arrow[r, "g"] \arrow[d, equal] & D[-1] \arrow[r] & 0 \\
 & 0 \arrow[r] & D \arrow[r, "\iota"] \arrow[d] & \text{cone}(g) \arrow[r, "\delta"] \arrow[d, equal] & C[-1] \arrow[r] & 0 & \\
0 \arrow[r] & C \arrow[r] & \text{cyl}(g) \arrow[r] & \text{cone}(g) \arrow[r] & 0 & &
\end{tikzcd}$$

which creates homology long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(C[-1]) \arrow[r] \arrow[d, equal] & H_{n+1}(D[-1]) \arrow[r, "\partial"] \arrow[d, equal] & H_n(B[-1]) \arrow[r] \arrow[d, "\psi_*"] & H_n(C[-1]) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & H_{n+1}(C[-1]) \arrow[r, "\partial"] \arrow[d, equal] & H_n(D) \arrow[r, "\iota_*"] \arrow[d, "\cong"] & H_n(\text{cone}(g)) \arrow[r] \arrow[d, equal] & H_n(C[-1]) \arrow[r, "\partial"] \arrow[d, equal] & \cdots \\
\cdots \arrow[r, "\partial"] & H_n(C) \arrow[r] & H_n(\text{cyl}(g)) \arrow[r] & H_n(\text{cone}(g)) \arrow[r, "\partial"] & H_{n-1}(C) \arrow[r] & \cdots
\end{tikzcd}$$

By the five lemma, ψ is a quasi-isomorphism. By the commutativity of the diagram at the top middle square,

$$\partial : H_n(D) \to H_{n-1}(B) = H_n(D) \xrightarrow{\iota_*} H_n(\text{cone}(g)) \cong H_{n-1}(B),$$

where ι_* is the map induced by the inclusion $D \overset{\iota}{\hookrightarrow} \text{cone}(g)$.

Exercise 1.5.8 Given a map $f : B \to C$ of complexes, let v denote the inclusion of C into cone(f). Show that there is a chain homotopy equivalence $\text{cone}(v) \to B[-1]$. This equivalence is the algebraic analogue of the topological fact that for any map $f : K \to L$ of (topological) cell complexes the cone of the inclusion $L \subseteq Cf$ is homotopy equivalent to the suspension of K.

First we explicitly construct cone(v). The map $v : C \hookrightarrow \text{cone}(f)$ is $v(c) = (0, c)$, so the complex is

$$\mathrm{cone}(v)_n = C_{n-1} \oplus \mathrm{cone}(f)_n = C_{n-1} \oplus B_{n-1} \oplus C_n$$

and the differential is $d_{\mathrm{cone}(v)} : C_{n-1} \oplus B_{n-1} \oplus C_n \to C_{n-2} \oplus B_{n-2} \oplus C_{n-1}$,

$$\begin{aligned} d_{\mathrm{cone}(v)}(c_{n-1}, b_{n-1}, c_n) &= (-d_C(c_{n-1}), d_{\mathrm{cone}(f)}(b_{n-1}, c_n) - v(c_{n-1})) \\ &= (-d_C(c_{n-1}), (-d_B(b_{n-1}), d_C(c_n) - f(b_{n-1})) - c_{n-1}) \\ &= (-d_C(c_{n-1}), -d_B(b_{n-1}), d_C(c_n) - f(b_{n-1}) - c_{n-1}); \end{aligned}$$

i.e.,

$$d_{\mathrm{cone}(v)} = \begin{bmatrix} -d_C & 0 & 0 \\ 0 & -d_B & 0 \\ -v & -f & d_C \end{bmatrix}.$$

Now we need to show that there exists a chain homotopy equivalence $\varphi : \mathrm{cone}(v) \to B[-1]$; i.e., that for such a φ there also exist a $\psi : B[-1] \to \mathrm{cone}(v)$ and chain contractions $\{s_n : \mathrm{cone}(v)_n \to \mathrm{cone}(v)_{n+1}\}$ and $\{t_n : B[-1]_n \to B[-1]_{n+1}\}$ such that

$$\begin{aligned} \mathrm{id}_{\mathrm{cone}(v)} - \psi\varphi &= ds + sd \text{ and} \\ \mathrm{id}_{B[-1]} - \varphi\psi &= dt + td. \end{aligned}$$

Define $\varphi : \mathrm{cone}(v) = C[-1] \oplus B[-1] \oplus C \to B[-1]$ to be $\varphi(c_{n-1}, b_{n-1}, c_n) = (-1)^n b_{n-1}$. Define $\psi : B[-1] \to \mathrm{cone}(v)$ to be $\psi(b_{n-1}) = ((-1)^{n+1} f(b_{n-1}), (-1)^n b_{n-1}, 0)$. Define

$\{s(c_{n-1}, b_{n-1}, c_n) = (-c_n, 0, 0)\}$ and $\{t(b_{n-1}) = 0\}$.[a] Then

$$\begin{aligned}
(\mathrm{id}_{\mathrm{cone}(v)} - \psi\varphi)(c_{n-1}, b_{n-1}, c_n) &= \mathrm{id}_{\mathrm{cone}(v)}(c_{n-1}, b_{n-1}, c_n) - \psi\varphi(c_{n-1}, b_{n-1}, c_n) \\
&= (c_{n-1}, b_{n-1}, c_n) - \psi((-1)^n b_{n-1}) \\
&= (c_{n-1}, b_{n-1}, c_n) - ((-1)^{n+1} f((-1)^n b_{n-1}), (-1)^n(-1)^n b_{n-1}, 0) \\
&= (c_{n-1}, b_{n-1}, c_n) - ((-1)^{2n+1} f(b_{n-1}), (-1)^{2n} b_{n-1}, 0) \\
&= (c_{n-1}, b_{n-1}, c_n) - (-f(b_{n-1}), b_{n-1}, 0) \\
&= (c_{n-1} + f(b_{n-1}), 0, c_n) \\
&= (d_C(c_n), 0, c_n) + (-d_C(c_n) + f(b_{n-1}) + c_{n-1}, 0, 0) \\
&= (-d_C(-c_n), 0, -(-c_n)) + (-(d_C(c_n) - f(b_{n-1}) - c_{n-1}), 0, 0) \\
&= d(-c_n, 0, 0) + s(-d_C(c_{n-1}), -d_B(b_{n-1}), d_C(c_n) - f(b_{n-1}) - c_{n-1}) \\
&= ds(c_{n-1}, b_{n-1}, c_n) + sd(c_{n-1}, b_{n-1}, c_n) \\
&= (ds + sd)(c_{n-1}, b_{n-1}, c_n),
\end{aligned}$$

and

$$\begin{aligned}
(\mathrm{id}_{B[-1]} - \varphi\psi)(b_{n-1}) &= b_{n-1} - \varphi\psi(b_{n-1}) \\
&= b_{n-1} - \varphi((-1)^{n+1} f(b_{n-1}), (-1)^n b_{n-1}, 0) \\
&= b_{n-1} - (-1)^n(-1)^n b_{n-1} \\
&= b_{n-1} - (-1)^{2n} b_{n-1} \\
&= b_{n-1} - b_{n-1} \\
&= 0 \\
\\
&= 0 + 0 \\
&= d(0) + t(db_{n-1}) \\
&= dt(b_{n-1}) + td(b_{n-1}) \\
&= (dt + td)(b_{n-1}),
\end{aligned}$$

so φ is a chain homotopy equivalence, as we wished to show.

[a]For future reference, first guess was $\varphi(c, b, c) = b$, $\psi(b) = (0, b, 0)$, but that didn't work nicely and I was worried it didn't depend on f. Then $\psi(b) = (fb, b, 0)$ was the guess, since that's the only way to get domains and codomains to line up nice. That was more promising, but the differential $d_{\text{cone}(v)}$ kept introducing nasty negatives. The final adjustments on φ, ψ, and s worked swimmingly.

Exercise 1.5.9 Let $f : B \to C$ be a morphism of chain complexes. Show that the natural maps $\ker(f)[-1] \xrightarrow{\alpha} \text{cone}(f) \xrightarrow{\beta} \text{coker}(f)$ give rise to a long exact sequence:

$$\cdots \xrightarrow{\partial} H_{n-1}(\ker(f)) \xrightarrow{\alpha} H_n(\text{cone}(f)) \xrightarrow{\beta} H_n(\text{coker}(f)) \xrightarrow{\partial} H_{n-2}(\ker(f)) \cdots .$$

First note that the natural maps α and β must be defined to be

$$\alpha(b_{n-1}) = (b_{n-1}, 0)$$
$$\beta(b_{n-1}, c_n) = c_n \text{ mod im } f.$$

Let $\iota : \text{im}(f) \hookrightarrow C$ be the inclusion map; then as $\text{coker}(f) \cong {}^{C}\!/_{\text{im}(f)}$, the following triangle commutes:

$$\begin{tikzcd}
 & & (f(b_{n-1}), c_n) \arrow[dddrr, mapsto, bend left] & & \\
 & & \begin{matrix}\text{cone}(\iota)\\ =\text{im} f[-1]\oplus C\end{matrix} \arrow[dr, "\varepsilon"] & & \\
 & \begin{matrix}\text{cone}(f)\\ =B[-1]\oplus C\end{matrix} \arrow[ur, "\gamma"] \arrow[rr, "\beta"] & & \begin{matrix}\text{coker}(f)\\ ={}^{C}\!/_{\text{im} f}\end{matrix} & \\
(b_{n-1}, c_n) \arrow[uuurr, mapsto, bend left] \arrow[rrrr, mapsto] & & & & c_n \text{ mod im } f
\end{tikzcd}$$

We can show that

$$0 \to \ker(f)[-1] \xrightarrow{\alpha} \text{cone}(f) \xrightarrow{\gamma} \text{cone}(\iota) \to 0$$

is a short exact sequence. Indeed, see that:

1. α is injective. If $\alpha(b_{n-1}) = (b_{n-1}, 0) = (0, 0)$, then $b_{n-1} = 0$.
2. γ is surjective. Given $(x, y) \in \text{cone}(\iota)_n$, $x \in \text{im} f_{n-1}$, so there exists $t \in B_{n-1}$ such that $f(t) = x$. Choose $(t, y) \in \text{cone}(f)$. Then $\gamma(t, y) = (f(t), y) = (x, y)$.
3. $\text{im}\,\alpha = \ker\gamma$. Let $(x, y) \in \text{im}\,\alpha$. Then $y = 0$ and $x \in \ker(f)$. Then observe that $\gamma(x, 0) = (f(x), 0) = (0, 0)$, so $(x, y) \in \ker\gamma$. On the other hand, if $(x, y) \in \ker\gamma$, then $\gamma(x, y) = (f(x), y) = (0, 0)$. So $y = 0$ and $x \in \ker(f)$. Thus $(x, y) \in \text{im}\,\alpha$.

By theorem 1.3.1, the above short exact sequence gives rise to a long exact sequence

$$\cdots \xrightarrow{\partial} H_{n-1}(\ker(f)) \xrightarrow{\alpha_*} H_n(\mathrm{cone}(f)) \xrightarrow{\gamma_*} H_n(\mathrm{cone}(\iota)) \xrightarrow{\partial} H_{n-2}(\ker(f)) \to \cdots$$

Since β factors through $\mathrm{cone}(\iota)$,

$$\begin{tikzcd} & H_n(\mathrm{cone}(\iota)) \arrow[dr, "\varepsilon_*"] & \\ H_n(\mathrm{cone}(f)) \arrow[ur, "\gamma_*"] \arrow[rr, "\beta_*"] & & H_n(\mathrm{coker}(f)) \end{tikzcd}$$

If we can show ε is a quasi-isomorphism, then we will complete the proof, because then

$$\begin{tikzcd} \cdots \arrow[r, "\partial"] & H_{n-1}(\ker(f)) \arrow[r, "\alpha_*"] & H_n(\mathrm{cone}(f)) \arrow[r, "\gamma_*"] \arrow[dr, "\beta_*"'] & H_n(\mathrm{cone}(\iota)) \arrow[r, "\partial"] \arrow[d, "\wr\ \varepsilon_*"] & H_{n-2}(\ker(f)) \arrow[r] & \cdots \\ & & & H_n(\mathrm{coker}(f)) & & \end{tikzcd}$$

By corollary 1.5.4, $\varepsilon : \mathrm{cone}(\iota) \to \mathrm{coker}(f)$ is a quasi-isomorphism if and only if $\mathrm{cone}(\varepsilon)$ is exact. We show that $\mathrm{cone}(\varepsilon)$ is exact. See that its construction is

$$\mathrm{cone}(\varepsilon)_n = \mathrm{cone}(\iota)_{n-1} \oplus \mathrm{coker}(f)_n = \mathrm{im}\, f_{n-2} \oplus C_{n-1} \oplus C_n/\mathrm{im}\, f_n$$

with differential

$$\begin{aligned} & d_{\mathrm{cone}(\varepsilon)}(x_{n-2}, c_{n-1}, c_n \bmod \mathrm{im}\, f_n) \\ &\quad = (-d_{\mathrm{cone}(\iota)}(x_{n-2}, c_{n-1}), d_{\mathrm{coker}(f)}(c_n \bmod \mathrm{im}\, f_n) - \varepsilon(x_{n-2}, c_{n-1})) \\ &\quad = (-(-d_{\mathrm{im}\, f}(x_{n-2}), d_C(c_{n-1}) - \iota(x_{n-2})), d_{\mathrm{coker}(f)}(c_n \bmod \mathrm{im}\, f_n) - \varepsilon(x_{n-2}, c_{n-1})) \\ &\quad = (d_{\mathrm{im}\, f}(x_{n-2}), -d_C(c_{n-1}) + x_{n-2}, d_{\mathrm{coker}(f)}(c_n \bmod \mathrm{im}\, f_n) - c_{n-1} \bmod \mathrm{im}\, f_{n-1})); \end{aligned}$$

i.e.,

$$\begin{tikzcd} \text{degree } n & \text{degree } n-1 \\ \mathrm{im}\, f_{n-2} \arrow[r, "+"] \arrow[ddr, "-"] & \mathrm{im}\, f_{n-3} \\ \bigoplus & \bigoplus \\ C_{n-1} \arrow[r, "+"] \arrow[ddr, "-"] & C_{n-2} \\ \bigoplus & \bigoplus \\ C_n/\mathrm{im}\, f_n \arrow[r, "+"] & C_{n-1}/\mathrm{im}\, f_{n-1} \end{tikzcd}$$

So we must show $\operatorname{im} d_{\operatorname{cone}(\varepsilon)} = \ker d_{\operatorname{cone}(\varepsilon)}$. Since $\operatorname{im} d_{\operatorname{cone}(\varepsilon)} \subseteq \ker d_{\operatorname{cone}(\varepsilon)}$ always, let $(q, r, [s]) \in \ker d_{\operatorname{cone}(\varepsilon)}$ at degree $n-1$; i.e., $q \in \operatorname{im} f_{n-3}$, $r \in C_{n-2}$, and $[s] \in \operatorname{coker}(f_{n-1})$. Then

$$\begin{aligned} d_{\operatorname{cone}(\varepsilon)}(q, r, [s]) = (d_{\operatorname{im} f}(q), -d_C(r) + q, d_{\operatorname{coker}(f)}([s]) - [r]) = (0, 0, 0);\ \text{i.e.,} \\ d_{\operatorname{im} f}(q) = 0, \\ -d_C(r) + q = 0,\ \text{and} \\ d_{\operatorname{coker}(f)}([s]) - [r] = 0. \end{aligned}$$

The above implies that

$$q = d_C(r) \in \operatorname{im} f,$$

and that

$$\begin{aligned} [r] &= d_{\operatorname{coker}(f)}([s]),\ \text{so} \\ r + fb' &= d_C(s + fb),\ \text{i.e.,} \\ r &= d_C(s + fb) - fb'. \end{aligned}$$

That means

$$q = d_C(r) = d_C(d_C(s + fb) - fb') = -d_C fb' = -f db',$$

so we may assume there exists a boundary $-db'$ such that $f(-db') = q$.

We need to show that for some $x_{n-2} \in \operatorname{im} f_{n-2}$, $c_{n-1} \in C_{n-1}$, and $[c_n] \in \operatorname{coker}(f)$,

$$(q, r, [s]) = (d_{\operatorname{im} f}(x_{n-2}), -d_C(c_{n-1}) + x_{n-2}, d_{\operatorname{coker}(f)}([c_n]) - [c_{n-1}]).$$

Choose $x_{n-2} = -fb'$. Then

$$d_{\operatorname{im} f}(x_{n-2}) = d(-fb') = f(-db') = q.$$

Choose $c_{n-1} = -s - fb$. Then

$$-d_C(c_{n-1}) + x_{n-2} = -d_C(-s - fb) - fb' = d_C(s + fb) - fb' = r.$$

Choose $c_n = 0$. Then

$$d_{\mathrm{coker}(f)}([c_n]) - [c_{n-1}] = d([0]) - [-s - fb] = [s].$$

Therefore, $(q, r, [s]) \in \operatorname{im} d_{\mathrm{cone}(\varepsilon)}$, and thus $\mathrm{cone}(\varepsilon)$ is exact, and ε is a quasi-isomorphism, as we yearned to demonstrate.

Exercise 1.5.10 Let C and C' be split complexes, with splitting maps s, s'. If $f : C \to C'$ is a morphism, show that $\sigma(c, c') = (-s(c), s'(c') - s'fs(c))$ defines a splitting of $\mathrm{cone}(f)$ if and only if the map $f_* : H_*(C) \to H_*(C')$ is zero.

Recall $\mathrm{cone}(f) = C[-1] \oplus C'$ and $d_{\mathrm{cone}(f)}(c, c') = (-dc, dc' - fc)$. We compute $d_{\mathrm{cone}(f)} \sigma d_{\mathrm{cone}(f)}$. See that

$$\begin{aligned}
d\sigma d(c, c') &= d\sigma(-dc, dc' - fc) \\
&= d(-s(-dc), s'(dc' - fc) - s'fs(-dc)) \\
&= d(sdc, s'dc' - s'fc + s'fsdc) \\
&= (-d(sdc), d(s'dc' - s'fc + s'fsdc) - f(sdc)) \\
&= (-dsdc, ds'dc' - ds'fc + ds'fsdc - fsdc).
\end{aligned}$$

As C and C' are split, $dsd = d$ and $ds'd = d$. So

$$(-dsdc, ds'dc' - ds'fc + ds'fsdc - fsdc) = (-dc, dc' - ds'fc + ds'fsdc - fsdc).$$

We need to show that

$$(-dc, dc' - ds'fc + ds'fsdc - fsdc) = (-dc, dc' - fc) = d_{\mathrm{cone}(f)}(c, c')$$

if and only if $f_* = 0$. That requires that

$$-ds'fc + ds'fsdc - fsdc = -fc.$$

Meanwhile, note that from 1.5.2,

$$\cdots \to H_{n+1}(\mathrm{cone}(f)) \to H_n(C) \xrightarrow{f_*} H_n(C') \to H_n(\mathrm{cone}(f)) \to \cdots$$

is long exact, so $H_n(C) \xrightarrow{f_*} H_n(C')$ is zero if and only if

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & H_{n+1}(\mathrm{cone}(f)) & \longrightarrow & H_n(C) & \xrightarrow{f_*} & H_n(C') & \longrightarrow & H_n(\mathrm{cone}(f)) & \longrightarrow & H_{n-1}(C) & \longrightarrow \cdots \\
& \| & & \| & & \downarrow & & \| & & \| & \\
\cdots \longrightarrow & H_{n+1}(\mathrm{cone}(f)) & \longrightarrow & H_n(C) & \xrightarrow{0} & 0 & \longrightarrow & H_n(\mathrm{cone}(f)) & \longrightarrow & H_{n-1}(C) & \longrightarrow \cdots
\end{array}$$

By the five lemma, $H_n(C') \cong 0$. Thus, C' is split exact, and so by exercise 1.4.3, this is the case if and only if $\mathrm{id}_{C'} = ds' + s'd$.

Returning, we need to show that

$$f = ds'f - ds'fsd + fsd$$

if and only if $f_* = 0$, which, by above, is the case if and only if $\mathrm{id} = ds' + s'd$. So see that

$$\begin{aligned}
ds'f - ds'fsd + fsd &= ds'f - ds'fsd + (ds' + s'd)fsd \\
&= ds'f - ds'fsd + ds'fsd + s'dfsd \\
&= ds'f + s'dfsd.
\end{aligned}$$

Since f is a chain map, $s'dfsd = s'fdsd$, and since $dsd = d$, $s'fdsd = s'fd$. Since f is a chain map, $s'fd = s'df$. Finally,

$$ds'f + s'df = (ds' + s'd)f = \mathrm{id}\, f = f,$$

and the result is shown.

1.6 More on Abelian Categories

We have already seen that R-**mod** is an abelian category for every associative ring R. In this section we expand our repertoire of abelian categories to include functor categories and sheaves. We also introduce the notions of left exact and right exact functors, which will form the heart of the next chapter. We give the Yoneda embedding of an additive category, which is exact and fully faithful, and use it to sketch a proof of the following result, which has already been used. Recall that a category is called *small* if its class of objects is in fact a set.

Freyd-Mitchell Embedding Theorem 1.6.1 (1964) *If $\mathcal{A}$ is a small abelian category, then there is a ring R and an exact, fully faithful functor from $\mathcal{A}$ into R-**mod**, which embeds $\mathcal{A}$ as a full subcategory in the sense that* $\mathrm{Hom}_{\mathcal{A}}(M,N) \cong \mathrm{Hom}_R(M,N)$.

We begin to prepare for this result by introducing some examples of abelian categories. The following criterion, whose proof we leave to the reader, is frequently useful:

Lemma 1.6.2 *Let $\mathcal{C} \subset \mathcal{A}$ be a full subcategory of an abelian category $\mathcal{A}$.*

1. *$\mathcal{C}$ is additive $\iff$ $0 \in \mathcal{C}$, and $\mathcal{C}$ is closed under $\oplus$.*
2. *$\mathcal{C}$ is abelian and $\mathcal{C} \subset \mathcal{A}$ is exact $\iff$ $\mathcal{C}$ is additive, and $\mathcal{C}$ is closed under* ker *and* coker.

Examples 1.6.3

1. Inside R-**mod**, the finitely generated R-modules form an additive category, which is abelian if and only if R is noetherian.
2. Inside **Ab**, the torsionfree groups form an additive category, while the p-groups form an abelian category. (A is a p-group if $(\forall a \in A)$ some $p^n a = 0$.) Finite p-groups also form an abelian category. The category $\left(\mathbf{Z}/p\right)$-**mod** of vector spaces over the field $\mathbf{Z}/p$ is also a full subcategory of **Ab**.

Functor Categories 1.6.4 Let C be any category, $\mathcal{A}$ an abelian category. The *functor category* $\mathcal{A}^C$ is the abelian category whose objects are functors $F : C \to \mathcal{A}$. The maps in $\mathcal{A}^C$ are natural transformations. Here are some relevant examples:

1. If C is the discrete category of integers, $\mathbf{Ab}^C$ contains the abelian category of *graded abelian groups* as a full subcategory.
2. If C is the poset category of integers $(\cdots \to n \to (n+1) \to \cdots)$ then the abelian category $\mathbf{Ch}(\mathcal{A})$ of cochain complexes if a full subcategory of $\mathcal{A}^C$.
3. If R is a ring considered as a one-object category, then R-**mod** is the full subcategory of all additive functors in $\mathbf{Ab}^R$.
4. Let X be a topological space, and $\mathcal{U}$ the poset of open subsets of X. A contravariant functor F from $\mathcal{U}$ to $\mathcal{A}$ such that $F(\emptyset) = \{0\}$ is called a *presheaf* on X with values in $\mathcal{A}$, and the presheaves are the objects of the abelian category $\mathcal{A}^{\mathcal{U}^{op}} = \mathrm{Presheaves}(X)$.

A typical example of a presheaf with values in **R-mod** is given by $C^0(U) = \{$continuous functions $f : U \to \mathbf{R}\}$. If $U \subset V$ the maps $C^0(V) \to C^0(U)$ are given by restricting the domain of a function from V to U. In fact, C^0 is a sheaf:

Definition 1.6.5 (Sheaves) A *sheaf* on X (with values in $\mathcal{A}$) is a presheaf F satisfying the

> *Sheaf Axiom.* Let $\{U_i\}$ be an open covering of an open subset U of X. If $\{f_i \in F(U_i)\}$ are such that each f_i and f_j agree in $F(U_i \cap U_j)$, then there is a unique $f \in F(U)$ that maps to every f_i under $F(U) \to F(U_i)$.

Note that the uniqueness of f is equivalent to the assertion that if $f \in F(U)$ vanishes in every $F(U_i)$, then $f = 0$. In fancy (element-free) language, the sheaf axiom states that for every covering $\{U_i\}$ of every open U the following sequence is exact:

$$0 \to F(U) \to \prod F(U_i) \xrightarrow{\text{diff}} \prod_{i<j} F(U_i \cap U_j).$$

Exercise 1.6.1 Let M be a smooth manifold. For each open U in M, let $C^\infty(U)$ be the set of smooth functions from U to $\mathbf{R}$. Show that C^∞ is a sheaf on M.

Exercise 1.6.2 (Constant sheaves) Let A be any abelian group. For every open subset U of X, let $A(U)$ denote the set of continuous maps from U to the discrete topological space A. Show that A is a sheaf on X.

The category Sheaves(X) of sheaves forms an abelian category contained in Presheaves(X), but it is not an abelian subcategory; cokernels in Sheaves(X) are different from cokernels in Presheaves(X). This difference gives rise to sheaf cohomology (Chapter 2, section 2.6). The following example lies at the heart of the subject. For any space X, let $\mathcal{O}$ (resp. $\mathcal{O}^*$) be the sheaf such that $\mathcal{O}(U)$ (resp. $\mathcal{O}^*(U)$) is the group of continuous maps from U into $\mathbf{C}$ (resp. $\mathbf{C}^*$). Then there is short exact sequence of sheaves:

$$0 \to \mathbf{Z} \xrightarrow{2\pi i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0.$$

When X is the space $\mathbf{C}^*$, this sequence is not exact in Presheaves(X) because the exponential map from $\mathbf{C} = \mathcal{O}(X)$ to $\mathcal{O}^*(X)$ is not onto; the cokernel is $\mathbf{Z} = H^1(X, \mathbf{Z})$, generated by the global unit $\frac{1}{z}$. In effect, there is no global logarithm function on X, and the contour integral $\frac{1}{2\pi i} \oint \frac{f'(z)}{f(z)} dz$ gives the image of $f(z)$ in the cokernel.

Definition 1.6.6 Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. F is called *left exact* (resp. *right exact*) if for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, the sequence $0 \to F(A) \to F(B) \to F(C)$ (resp. $F(A) \to F(B) \to F(C) \to 0$) is exact in $\mathcal{B}$. F is called *exact* if it is both left and right exact, that is, if it preserves exact sequences. A contravariant functor F is called left exact (resp. right exact, resp. exact) if the corresponding covariant functor $F' : \mathcal{A}^{op} \to \mathcal{B}$ is left exact (resp. ...).

Example 1.6.7 The inclusion of Sheaves(X) into Presheaves(X) is a left exact functor. There is also an exact functor Presheaves(X) $\to$ Sheaves(X), called "sheafification." (See 2.6.5; the sheafification functor is left adjoint to the inclusion.)

Exercise 1.6.3 Show that the above definitions are equivalent to the following, which are often given as the definitions. (See [Rot], for example.) A (covariant) functor F is left exact (resp. right exact) if exactness of the sequence

$$0 \to A \to B \to C \qquad\qquad \text{(resp. } A \to B \to C \to 0\text{)}$$

implies exactness of the sequence

$$0 \to FA \to FB \to FC \qquad\qquad \text{(resp. } FA \to FB \to FC \to 0\text{).}$$

Proposition 1.6.8 *Let $\mathcal{A}$ be an abelian category. Then* $\mathrm{Hom}_{\mathcal{A}}(M,-)$ *is a left exact functor from* $\mathcal{A}$ *to* $\mathbf{Ab}$ *for every M in $\mathcal{A}$. That is, given an exact sequence* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *in $\mathcal{A}$, the following sequence of abelian groups is also exact:*

$$0 \to \mathrm{Hom}(M,A) \xrightarrow{f_*} \mathrm{Hom}(M,B) \xrightarrow{g_*} \mathrm{Hom}(M,C).$$

Proof. If $\alpha \in \mathrm{Hom}(M,A)$ then $f_*\alpha = f \circ \alpha$; if this is zero, then α must be zero since f is monic. Hence f_* is monic. Since $g \circ f = 0$, we have $g_*f_*(\alpha) = g \circ f \circ \alpha = 0$, so $g_*f_* = 0$. It remains to show that if $\beta \in \mathrm{Hom}(M,B)$ is such that $g_*\beta = g \circ \beta$ is zero, then $\beta = f \circ \alpha$ for some α. But if $g \circ \beta = 0$, then $\beta(M) \subseteq f(A)$, so β factors through A. $\square$

Corollary 1.6.9 $\mathrm{Hom}_{\mathcal{A}}(-,M)$ *is a left exact contravariant functor.*

Proof. $\mathrm{Hom}_{\mathcal{A}}(A,M) = \mathrm{Hom}_{\mathcal{A}^{op}}(M,A)$. $\square$

Yoneda Embedding 1.6.10 Every additive category $\mathcal{A}$ can be embedded in the abelian category $\mathbf{Ab}^{\mathcal{A}^{op}}$ by the functor h sending A to $h_A = \mathrm{Hom}_{\mathcal{A}}(-,A)$. Since each $\mathrm{Hom}_{\mathcal{A}}(M,-)$ is left exact, h is a left exact functor. Since the functors h_A are left exact, the Yoneda embedding actually lands in the abelian subcategory $\mathcal{L}$ of all left exact contravariant functors from $\mathcal{A}$ to $\mathbf{Ab}$ whenever $\mathcal{A}$ is an abelian category.

Yoneda Lemma 1.6.11 *The Yoneda embedding h reflects exactness. That is, a sequence* $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ *in $\mathcal{A}$ is exact, provided that for every M in $\mathcal{A}$ the following sequence is exact:*

$$\mathrm{Hom}_{\mathcal{A}}(M,A) \xrightarrow{\alpha_*} \mathrm{Hom}_{\mathcal{A}}(M,B) \xrightarrow{\beta_*} \mathrm{Hom}_{\mathcal{A}}(M,C).$$

Proof. Taking $M = A$, we see that $\beta\alpha = \beta^*\alpha^*(\mathrm{id}_A) = 0$. Taking $M = \ker(\beta)$, we see that the inclusion $\iota : \ker(\beta) \to B$ satisfies $\beta^*(\iota) = \beta\iota = 0$. Hence there is a $\sigma \in \mathrm{Hom}(M,A)$ with $\iota = \alpha^*(\sigma) = \alpha\sigma$, so that $\ker(\beta) = \mathrm{im}(\iota) \subseteq \mathrm{im}(\alpha)$. $\square$

We now sketch a proof of the Freyd-Mitchell Embedding Theorem 1.6.1; details may be found in [Freyd] or [Swan, pp. 14-22]. Consider the failure of the Yoneda embedding $h : \mathcal{A} \to \mathbf{Ab}^{\mathcal{A}^{op}}$ to be exact: if $0 \to A \to B \to C \to 0$ is exact in $\mathcal{A}$ and $M \in \mathcal{A}$, then define the abelian group $W(M)$ by exactness of

$$0 \to \mathrm{Hom}_{\mathcal{A}}(M,A) \to \mathrm{Hom}_{\mathcal{A}}(M,B) \to \mathrm{Hom}_{\mathcal{A}}(M,C) \to W(M) \to 0.$$

In general $W(M) \neq 0$, and there is a short exact sequence of functors:

$$0 \to h_A \to h_B \to h_C \to W \to 0. \qquad (*)$$

W is an example of a *weakly effaceable functor*, that is, a functor such that for all $M \in \mathcal{A}$ and $x \in W(M)$ there is a surjection $P \to M$ in $\mathcal{A}$ so that the map $W(M) \to W(P)$ sends x to zero. (To see this, take P to be the pullback $M \times_C B$, where $M \to C$ represents x, and note that $P \to C$ factors through B. Next (see *loc. cit.*), one proves:

Proposition 1.6.12 *If $\mathcal{A}$ is small, the subcategory $\mathcal{W}$ of weakly effaceable functors is a localizing subcategory of* $\mathbf{Ab}^{\mathcal{A}^{op}}$ *whose quotient category is $\mathcal{L}$. That is, there is an exact "reflection" functor R from* $\mathbf{Ab}^{\mathcal{A}^{op}}$ *to $\mathcal{L}$ such that $R(L) = L$ for every left exact L and $R(W) \cong 0$ iff W is weakly effaceable.*

Remark Cokernels in $\mathcal{L}$ are different from cokernels in $\mathbf{Ab}^{\mathcal{A}^{op}}$, so the inclusion $\mathcal{L} \subset \mathbf{Ab}^{\mathcal{A}^{op}}$ is not exact, merely left exact. To see this, apply the reflection R to $(*)$. Since $R(h_A) = h_A$ and $R(W) \cong 0$, we see that

$$0 \to h_A \to h_B \to h_C \to 0$$

is an exact sequence in $\mathcal{L}$, but not in $\mathbf{Ab}^{\mathcal{A}^{op}}$

Corollary 1.6.13 *The Yoneda embedding $h : \mathcal{A} \to \mathcal{L}$ is exact and fully faithful.*

Finally, one observes that the category $\mathcal{L}$ has arbitrary coproducts and has a faithfully projective object P. By a result of Gabriel and Mitchell [Freyd, p. 106], every small full abelian subcategory of $\mathcal{L}$ is equivalent to a full abelian subcategory of the category R-**mod** of modules over the ring $R = \mathrm{Hom}_{\mathcal{L}}(P, P)$. This finishes the proof of the Embedding Theorem.

Example 1.6.14 The abelian category of graded R-modules may be thought of as the full subcategory of $(\pi_{i\in\mathbf{Z}}R)$-modules of the form $\oplus_{i\in\mathbf{Z}}M_i$. The abelian category of chain complexes of R-modules may be embedded in S-**mod**, where

$$S = (\prod_{i\in\mathbf{Z}} R)[d] \Big/ (d^2 = 0, \{dr = rd\}_{r\in R}, \{de_i = e_{i-1}d\}_{i\in\mathbf{Z}}).$$

Here $e_i : \prod R \to R \to \prod R$ is the i^{th} coordinate projection.

2.1 δ-Functors

The right context in which to view derived functors, according to Grothendieck [Tohoku], is that of δ-functors between two abelian categories $\mathcal{A}$ and $\mathcal{B}$.

Definition 2.1.1 A (covariant) *homological* (resp. *cohomological*) *δ-functor* between $\mathcal{A}$ and $\mathcal{B}$ is a collection of additive functors $T_n : \mathcal{A} \to \mathcal{B}$ (resp. $T^n : \mathcal{A} \to \mathcal{B}$) for $n \geq 0$, together with morphisms

$$\delta_n : T_n(C) \to T_{n-1}(A)$$
$$(\text{resp. } \delta^n : T^n(C) \to T^{n+1}(A))$$

defined for each short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$. Here we make the convention that $T^n = T_n = 0$ for $n < 0$. These two conditions are imposed:

1. For each short exact sequence as above, there is a long exact sequence

   $$\cdots T_{n+1}(C) \xrightarrow{\delta} T_n(A) \to T_n(B) \to T_n(C) \xrightarrow{\delta} T_{n-1}(A) \cdots$$

   (resp.

   $$\cdots T^{n-1}(C) \xrightarrow{\delta} T^n(A) \to T^n(B) \to T^n(C) \xrightarrow{\delta} T^{n+1}(A) \cdots).$$

   In particular, T_0 is right exact, and T^0 is left exact.

2. For each morphism of short exact sequences from $0 \to A' \to B' \to C' \to 0$ to $0 \to A \to B \to C \to 0$, the δ's give a commutative diagram

$$\begin{array}{ccccccc} T_n(C') & \xrightarrow{\delta} & T_{n-1}(A') & & T^n(C') & \xrightarrow{\delta} & T^{n+1}(A') \\ \downarrow & & \downarrow & \text{resp.} & \downarrow & & \downarrow \\ T_n(C) & \xrightarrow{\delta} & T_{n-1}(A) & & T^n(C) & \xrightarrow{\delta} & T^{n+1}(A). \end{array}$$

Example 2.1.2 Homology gives a homological δ-functor H_* from $\mathbf{Ch}_{\geq 0}(\mathcal{A})$ to $\mathcal{A}$; cohomology gives a cohomological δ-functor H^* from $\mathbf{Ch}^{\geq}(\mathcal{A})$ to $\mathcal{A}$.

Exercise 2.1.1 Let $\mathcal{S}$ be the category of short exact sequences

$$0 \to A \to B \to C \to 0 \qquad (*)$$

in $\mathcal{A}$. Show that δ_i is a natural transformation from the functor sending $(*)$ to $T_i(C)$ to the functor sending $(*)$ to $T_{i-1}(A)$.

A natural transformation ν from a functor $F : \mathcal{C} \to \mathcal{D}$ to a functor $G : \mathcal{C} \to \mathcal{D}$ is a family of morphisms satisfying

1. ν must associate to each object X in $\mathcal{C}$ a morphism $\nu_X : F(X) \to G(X)$ between objects in $\mathcal{D}$, and

2. for all $f : X \to Y$ in $\mathcal{C}$ we get $\nu_Y \circ F(f) = G(f) \circ \nu_X$; i.e.,

$$\begin{array}{ccc} F(X) & \xrightarrow{\nu_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\nu_Y]{} & G(Y). \end{array}$$

Let F be the functor sending $(*)$ to $T_i(C)$ and G be the functor sending $(*)$ to $T_{i-1}(A)$. To see that δ_i from F to G satisfies these two conditions, see that

1. to each short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{S}$, δ_i associates a morphism $T_i(C) \to T_{i-1}(A)$. Indeed, it's δ_i.
2. if I have a morphism f of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

we need to show that $\delta_i \circ (T_i(C) \xrightarrow{F(f)} T_i(C')) = (T_{i-1}(A) \xrightarrow{G(f)} T_{i-1}(A')) \circ \delta_i$; i.e., that

$$\begin{array}{ccc} T_i(C) & \xrightarrow{\delta_i} & T_{i-1}(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ T_i(C') & \xrightarrow[\delta_i]{} & T_{i-1}(A'). \end{array}$$

But there is nothing to show, as this is the case by the second condition on the definition of δ-functor.

Example 2.1.3 (p-torsion) If p is an integer, the functors $T_0(A) = A/pA$ and

$$T_1(A) = {}_pA \equiv \{a \in A \mid pa = 0\}$$

fit together to form a homological δ-functor, or a cohomological δ-functor (with $T^0 = T_1$ and $T^1 = T_0$) from **Ab** to **Ab**. To see this, apply the Snake Lemma to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

to get the exact sequence

$$0 \to {}_pA \to {}_pB \to {}_pC \xrightarrow{\delta} A/pA \to B/pB \to C/pC \to 0.$$

Generalization The same proof shows that if r is any element in a ring R, then $T_0(M) = M/rM$ and $T_1(M) = {}_rM$ fit together to form a homological δ-functor (or cohomological δ-functor, if that is one's taste) from R-**mod** to **Ab**.

Vista We will see in 2.6.3 that $T_n(M) = \mathrm{Tor}_n^R\left(R/r, M\right)$ is also a homological δ-functor with $T_0(M) = M/rM$. If r is a left nonzerodivisor (meaning that ${}_rR = \{s \in R \mid rs = 0\}$ is zero), then in fact $\mathrm{Tor}_1^R\left(R/r, M\right) = {}_rM$ and $\mathrm{Tor}_n^R\left(R/r, M\right) = 0$ for $n \geq 2$; see 3.1.7. However, in general ${}_rR \neq 0$, while $\mathrm{Tor}_1^R\left(R/r, R\right) = 0$, so they aren't the same; $\mathrm{Tor}_1^R\left(M, {}^{R}\!/_{r}\right)$ is the quotient of ${}_rM$ by the submodule $({}_rR)M$ generated by $\{sm \mid rs = 0, s \in R, m \in M\}$. The Tor_n will be *universal* δ-functors in a sense that we shall now make precise.

Definition 2.1.4 A *morphism* $S \to T$ of δ-functors is a system of natural transformations $S_n \to T_n$ (resp. $S^n \to T^n$) that commute with δ. This is fancy language for the assertion that there is a commutative ladder diagram connecting the long exact sequences for S and T associated to any short exact sequence in $\mathcal{A}$.

A homological δ-functor T is *universal* if, given any other δ-functor S and a natural transformation $f_0 : S_0 \to T_0$, there exists a unique morphism $\{f_n : S_n \to T_n\}$ of δ-functors that extends f_0.

A cohomological δ-functor T is *universal* if, given S and $f^0 : T^0 \to S^0$, there exists a unique morphism $T \to S$ of δ-functors extending f^0.

Example 2.1.5 We will see in section 2.4 that homology $H_* : \mathbf{Ch}_{\geq 0}(\mathcal{A}) \to \mathcal{A}$ and cohomology $H^* : \mathbf{Ch}^{\geq 0}(\mathcal{A}) \to \mathcal{A}$ are universal δ-functors.

Exercise 2.1.2 If $F : \mathcal{A} \to \mathcal{B}$ is an exact functor, show that $T_0 = F$ and $T_n = 0$ for $n \neq 0$ defines a universal δ-functor (of both homological and cohomological type).

An exact functor takes short exact sequences to short exact sequences; i.e., assuming F is covariant, $0 \to A \to B \to C \to 0$ exact implies $0 \to F(A) \to F(B) \to F(C) \to 0$ is exact. We show that T is a δ-functor first. See that for $\delta = \{\delta_n = 0\}$, we have for condition 1 a long exact sequence

$$\begin{tikzcd}
 & & \ddots \arrow[dll, "\delta"'] \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[dll, "\delta"'] \\
F(A) \arrow[r] & F(B) \arrow[r] & F(C) \arrow[dll, "\delta"'] \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & 0.
\end{tikzcd}$$

The only place to check exactness is at $F(C) \xrightarrow{\delta} A$. See that $\ker(A \to B) = \operatorname{im}\delta = 0$, since $A \to B$ is injective, and $\operatorname{im}(F(B) \to F(C)) = \ker\delta = F(C)$, since $F(B) \to F(C)$ is surjective.

For condition 2, if

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0,
\end{tikzcd}$$

then

$$\begin{tikzcd}
T_i(C) \arrow[r, "\delta"] \arrow[d] & T_{i-1}(A) \arrow[d] \\
T_i(C') \arrow[r, "\delta"] & T_{i-1}(A')
\end{tikzcd}$$

obviously commutes, since δ are all 0.

Now, we show that T is universal. Suppose S is another δ-functor such that $f_0 : S_0 \to T_0$ is a natural transformation; i.e., given a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, we have the start of the ladder diagram

$$\begin{tikzcd}
\cdots \arrow[r] & S_2(C) \arrow[r] & S_1(A) \arrow[r] & S_1(B) \arrow[r] & S_1(C) \arrow[r] & S_0(A) \arrow[r] \arrow[d, "f_0"] & S_0(B) \arrow[r] \arrow[d, "f_0"] & S_0(C) \arrow[r] \arrow[d, "f_0"] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & F(A) \arrow[r] \arrow[d, equal] & F(B) \arrow[r] \arrow[d, equal] & F(C) \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & T_2(C) \arrow[r] & T_1(A) \arrow[r] & T_1(B) \arrow[r] & T_1(C) \arrow[r] & T_0(A) \arrow[r] & T_0(B) \arrow[r] & T_0(C) \arrow[r] & 0.
\end{tikzcd}$$

But clearly the unique $\{f_n : S_n \to T_n\}$ must be all zero maps.

To see it is of cohomological type, quickly observe that

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[dll, rounded corners, "\delta"'] \\
& F(A) \arrow[r] & F(B) \arrow[r] & F(C) \arrow[dll, rounded corners, "\delta"'] \\
& 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[dll, rounded corners, "\delta"'] \\
& \ddots & &
\end{tikzcd}$$

with $\{\delta_n = 0\}$ is long exact,

$$\begin{tikzcd}
T^i(C) \arrow[r, "\delta"] \arrow[d] & T^{i+1}(A) \arrow[d] \\
T^i(C) \arrow[r, "\delta"] & T^{i+1}(A')
\end{tikzcd}$$

commutes given a morphism of short exact sequences, and

$$\begin{tikzcd}
0 \arrow[r] & T^0(A) \arrow[r] \arrow[d, equal] & T^0(B) \arrow[r] \arrow[d, equal] & T^0(C) \arrow[r] \arrow[d, equal] & T^1(A) \arrow[r] \arrow[d, equal] & T^1(B) \arrow[r] \arrow[d, equal] & T^1(C) \arrow[r] \arrow[d, equal] & T^2(A) \arrow[r] \arrow[d, equal] & \cdots \\
0 \arrow[r] & F(A) \arrow[r] \arrow[d, "f^0"] & F(B) \arrow[r] \arrow[d, "f^0"] & F(C) \arrow[r] \arrow[d, "f^0"] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
0 \arrow[r] & S^0(A) \arrow[r] & S^0(B) \arrow[r] & S^0(C) \arrow[r] & S^1(A) \arrow[r] & S^1(B) \arrow[r] & S^1(C) \arrow[r] & S^2(A) \arrow[r] & \cdots
\end{tikzcd}$$

only extends if f^n are all zero.

Remark If $F : \mathcal{A} \to \mathcal{B}$ is an additive functor, then we can ask if there is *any* δ-functor T (universal or not) such that $T_0 = F$ (resp. $T^0 = F$). One obvious obstruction is that T_0 must be right exact (resp. T^0 must be left exact). By definition, however, we see that there is at most one (up to isomorphism) *universal* δ-functor T with $T_0 = F$ (resp. $T^0 = F$). If a universal T exists, the T_n are sometimes called the *left satellite functors* of F (resp. the T^n are called the *right satellite functors* of F). This terminology is due to the pervasive influence of the book [CE].

We will see that derived functors, when they exist, are indeed universal δ-functors. For this we need the concept of projective and injective resolutions.

2.2 Projective Resolutions

An object P in an abelian category $\mathcal{A}$ is *projective* if it satisfies the following universal lifting property: Given a surjection $g : B \to C$ and a map $\gamma : P \to C$, there is at least one map $\beta : P \to B$ such that $\gamma = g \circ \beta$.

$$\begin{tikzcd} & P \arrow[dl, "\exists\beta"'] \arrow[d, "\gamma"] & \\ B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

We shall be mostly concerned with the special case of projective modules ($\mathcal{A}$ being the category **mod-**R). The notion of projective module first appeared in the book [CE]. It is easy to see that free R-modules are projective (lift a basis). Clearly, direct summands of free modules are also projective modules.

Proposition 2.2.1 *An R-module is projective iff it is a direct summand of a free R-module.*

Proof. Letting $F(A)$ be the free R-module on the set underlying an R-module A, we see that for every R-module A there is a surjection $\pi : F(A) \to A$. If A is a projective R-module, the universal lifting property yields a map $i : A \to F(A)$ so that $\pi i = 1_A$, that is, A is a direct summand of the free module $F(A)$. $\square$

Example 2.2.2 Over many nice rings ($\mathbf{Z}$, fields, division rings, $\cdots$) every projective module is in fact a free module. Here are two examples to show that this is not always the case:

1. If $R = R_1 \times R_2$, then $P = R_1 \times 0$ and $0 \times R_2$ are projective because their sum is R. P is not free because $(0, 1)P = 0$. This is true, for example, when R is the ring $\mathbf{Z}/6 = \mathbf{Z}/2 \times \mathbf{Z}/3$.
2. Consider the ring $R = M_n(F)$ of $n \times n$ matrices over a field F, acting on the left column vector space $V = F^n$. As a left R-module, R is the direct sum of its columns, each of which is the left R-module V. Hence $R \cong V \oplus \cdots \oplus V$, and V is a projective R-module. Since any free R-module would have dimension dn^2 over F for some cardinal number d, and $\dim_F(V) = n$, V cannot possibly be free over R.

Remark The category $\mathcal{A}$ of finite abelian groups is an example of an abelian category that has *no* projective objects except 0. We say that $\mathcal{A}$ *has enough projectives* if for every object A of $\mathcal{A}$ there is a surjection $P \to A$ with P projective.

Here is another characterization of projective objects in $\mathcal{A}$:

Lemma 2.2.3 *M is projective iff $\mathrm{Hom}_{\mathcal{A}}(M, -)$ is an exact functor. That is, iff the sequence of groups*

$$0 \to \mathrm{Hom}(M, A) \to \mathrm{Hom}(M, B) \xrightarrow{g_*} \mathrm{Hom}(M, C) \to 0$$

is exact for every exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$.

Proof. Suppose that $\mathrm{Hom}(M, -)$ is exact and that we are given a surjection $g : B \to C$ and a map $\gamma : M \to C$. We can lift $\gamma \in \mathrm{Hom}(M, C)$ to $\beta \in \mathrm{Hom}(M, B)$ such that $\gamma = g_*\beta = g \circ \beta$ because g_* is onto. Thus M has the universal lifting property, that is, it is projective. Conversely, suppose M is projective. In order to show that $\mathrm{Hom}(M, -)$ is exact, it suffices to show that g_* is onto for every short exact sequence as above. Given $\gamma \in \mathrm{Hom}(M, C)$, the universal lifting property of M gives $\beta \in \mathrm{Hom}(M, B)$ so that $\gamma = g \circ \beta = g_*(\beta)$, that is, g_* is onto. $\square$

A chain complex P in which each P_n is projective in $\mathcal{A}$ is called a *chain complex of projectives*. It need not be a projective object in **Ch**.

Exercise 2.2.1 Show that a chain complex P is a projective object in **Ch** if and only if it is a split exact complex of projectives. Their brutal truncations $\sigma_{\geq 0}P$ form the projective objects in $\mathbf{Ch}_{\geq 0}$. *Hint*: To see that P must be split exact, consider the surjection from $\operatorname{cone}(\mathrm{id}_P)$ to $P[-1]$. To see that split exact complexes are projective objects, consider the special case $0 \to P_1 \cong P_0 \to 0$.

First, suppose that P is a projective object; i.e., for all $B \twoheadrightarrow C$,

$$\begin{tikzcd} & P_\bullet \arrow[dl, dashed] \arrow[d] & \\ B_\bullet \arrow[r] & C_\bullet \arrow[r] & 0. \end{tikzcd}$$

By definition, these maps are defined on the level of degrees; i.e., for all i,

$$\begin{tikzcd} & P_i \arrow[dl, dashed] \arrow[d] & \\ B_i \arrow[r] & C_i \arrow[r] & 0. \end{tikzcd}$$

We show that P is a split exact complex of projectives. By definition, each P_i is projective. To see that $P_\bullet$ is split exact, we use exercise 1.4.3, and show that id_P is nulhomotopic. Since $\operatorname{cone}(P) \to P \to 0$, we have the diagram

$$\begin{tikzcd} & P_n \arrow[d, "\mathrm{id}"] & \\ P_{n-1} \oplus P_n \arrow[r] & P_n \arrow[r] & 0 \end{tikzcd}$$

Define $s : P \to \operatorname{cone}(P)$ to be the map guaranteed by projective-ness of P. Then for all n,

$$\begin{tikzcd} & P_n \arrow[dl, "s_n"'] \arrow[d, "\mathrm{id}"] & \\ P_{n-1} \oplus P_n \arrow[r] & P_n \arrow[r] & 0. \end{tikzcd}$$

By exercise 1.5.2, id_P is nulhomotopic if and only if id extends to a map $(*, \mathrm{id}) : \operatorname{cone}(P) \to P$; i.e.,

$$\begin{tikzcd} & \operatorname{cone}(P) \arrow[dr] & \\ P \arrow[ur] \arrow[rr, "\mathrm{id}"'] & & P. \end{tikzcd}$$

Since this is the case, id is nulhomotopic, and thus P is split exact, as desired.

SOMETHING IS WRONG HERE, because even though projectiveness guaranteed the existence of the map, given an arbitrary chain complex, you should be able to

$$\begin{tikzcd}
 & C_n \arrow[dl, "\iota_2"'] \arrow[d, "\mathrm{id}"] \\
C_{n-1} \oplus C_n \arrow[r, "\pi_2"'] & C_n
\end{tikzcd}$$

So what went wrong?

On the other hand, suppose P is a split exact complex of projective objects. We need to show that $P_\bullet$ itself is projective. Let $g : B_\bullet \to C_\bullet$ be a surjection, and assume $\gamma : P_\bullet \to C_\bullet$. We need to construct a map $\beta : P_\bullet \to B_\bullet$ such that $\gamma = g\beta$. Following the hint, we show that the problem can be reduced to the case where P_0, P_1 are projective and

$$0 \to P_1 \xrightarrow{\sim} P_0 \to 0$$

is exact. Since P is split exact, by exercise 1.4.2, $P_n \cong \ker(d_n) \oplus \mathrm{im}(d_n)$. Since P_n is projective, it is a direct summand of free modules, and thus $\ker(d_n)$ and $\mathrm{im}(d_n)$ must be projective too. Also since P is exact, $\mathrm{im}(d_n) = \ker(d_{n+1})$. Now consider the complex

$$Q(n) = 0 \to \mathrm{im}(d_n) \to \ker(d_{n+1}) \to 0.$$

Since $P_\bullet = \bigoplus_{n \in \mathbf{Z}} Q(n)$ (once you line up the degrees correctly), we have reduced to the case of the hint. Solving this problem, we will then explain how to pass through the direct sum.

So assume $0 \to P_1 \overset{d}{\xrightarrow{\sim}} P_0 \to 0$ is split exact with P_i projective, and that $g : B_\bullet \to C_\bullet$ is a surjection, and that $\gamma : P_\bullet \to C_\bullet$. We construct $\beta : P_\bullet \to B_\bullet$. Since P is zero outside of degrees 0 and 1, so is β. We get

$$\begin{tikzcd}
 & P_1 \arrow[dl, dashed, "\beta_1"'] \arrow[d, "\gamma_1"] & \\
B_1 \arrow[r, "g_1"'] & C_1 \arrow[r] & 0
\end{tikzcd}$$

by projective-ness of P_1, and let $\beta_0 : P_0 \to B_0$ be

$$P_0 \xrightarrow[\sim]{d^{-1}} P_1 \xrightarrow{\beta_1} B_1 \xrightarrow{d} B_0.$$

To confirm that this works, see that

$$\begin{aligned}
&g_0\beta_0 = g_0 d_B \beta_1 {d_P}^{-1} = d_C g_1 \beta_1 {d_P}^{-1} = d_C \gamma_1 {d_P}^{-1} = \gamma_0 d_P {d_P}^{-1} = \gamma_0, \text{ and} \\
&g_1\beta_1 = \gamma_1,
\end{aligned}$$

so $0 \to P_1 \to P_0 \to 0$ is a projective object.

To see that this proves the general case, observe that if we are given

$$\begin{tikzcd}
 & P \arrow[d, "\gamma"] & \\
B \arrow[r, "g"] & C \arrow[r] & 0,
\end{tikzcd}$$

then γ restricts to a map $\gamma(n) : Q(n) \to C$ where $\gamma = \sum_{n\in\mathbf{Z}} \gamma(n)$. Since we have shown that there exists $\beta(n) : Q(n) \to B$ such that $g\beta(n) = \gamma(n)$, we conclude that $\beta = \sum_{n\in\mathbf{Z}} \beta(n)$ must satisfy $g\beta = \gamma$, as desired.

Exercise 2.2.2 Use the previous exercise 2.2.1 to show that if $\mathcal{A}$ has enough projectives, then so does the category $\mathbf{Ch}(\mathcal{A})$ of chain complexes over $\mathcal{A}$.

If $\mathcal{A}$ has enough projectives, then for all objects A in $\mathcal{A}$, there exists a $P \to A \to 0$ with P projective. We need to show that $\mathbf{Ch}(\mathcal{A})$ has enough projectives; i.e., given

$$A_\bullet = \qquad \cdots \to A_{n+1} \to A_n \to A_{n-1} \to \cdots,$$

there exists a projective $P_\bullet$ (i.e., split exact complex with P_i projective, by above) such that $P_\bullet \to A_\bullet \to 0$. First, see that we can construct a complex of projective objects in $\mathcal{A}$ (not necessarily split exact yet):

For each n, we can construct a P_n since $\mathcal{A}$ has enough projectives:

$$\begin{tikzcd}
 & {} \arrow[d] & \\
P_{n+1} \arrow[r] & A_{n+1} \arrow[r] \arrow[d] & 0 \\
P_n \arrow[r] & A_n \arrow[r] \arrow[d] & 0 \\
P_{n-1} \arrow[r] & A_{n-1} \arrow[r] \arrow[d] & 0 \\
 & {} &
\end{tikzcd}$$

And we can construct $d_n : P_n \to P_{n-1}$ by using the fact that P_n is projective and

$$\begin{tikzcd} & P_n \arrow[d] \arrow[dl, dashed, "d_n"'] & \\ & A_n \arrow[d] & \\ P_{n-1} \arrow[r] & A_{n-1} \arrow[r] & 0. \end{tikzcd}$$

Now we need to use this complex to build a split exact complex, hence projective in $\mathbf{Ch}(\mathcal{A})$. To do this, consider cone$(P)[+1]$. Then cone$(P)[+1]$ is projective in $\mathbf{Ch}(\mathcal{A})$, since it's split exact and composed of direct sums of projectives, hence projectives. And we have the desired surjection, because for all n

$$P_n \oplus P_{n+1} \to A_n \to 0,$$

where the map is the surjection we constructed on the first coordinate, and in the second coordinate, $P_{n+1} \xrightarrow{d} P_n \to A_n$. (Do I need a $\pm$ for that map? I think so...)

Definition 2.2.4 Let M be an object of $\mathcal{A}$. A *left resolution* of M is a complex $P_\bullet$ with $P_i = 0$ for $i < 0$, together with a map $\varepsilon : P_0 \to M$ so that the augmented complex

$$\cdots \xrightarrow{d} P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{\varepsilon} M \to 0$$

is exact. It is a *projective resolution* if each P_i is projective.

Lemma 2.2.5 *Every R-module M has a projective resolution. More generally, if an abelian category $\mathcal{A}$ has enough projectives, then every object M in $\mathcal{A}$ has a projective resolution.*

$$\begin{tikzcd}
0 \arrow[dr] & & 0 & & 0 \arrow[dr] & & 0 & & & & \\
& M_3 \arrow[ur] \arrow[dr] & & & & M_1 \arrow[ur] \arrow[dr] & & & & & \\
\cdots \arrow[ur] \arrow[rr] & & P_3 \arrow[rr, "d"] \arrow[dr] & & P_2 \arrow[rr, "d"] \arrow[ur] & & P_1 \arrow[rr, "d"] \arrow[dr] & & P_0 \arrow[r, "\varepsilon_0"] & M \arrow[r] & 0 \\
& & & M_2 \arrow[ur] \arrow[dr] & & & & M_0 \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Figure 2.1. Forming a resolution by splicing.

Proof. Choose a projective P_0 and a surjection $\varepsilon_0 : P_0 \to M$, and set $M_0 = \ker(\varepsilon_0)$. Inductively, given a module M_{n-1}, we choose a projective P_n and a surjection $\varepsilon_n : P_n \to M_{n-1}$. Set $M_n = \ker(\varepsilon_n)$, and let d_n be the composite $P_n \to M_{n-1} \to P_{n-1}$. Since $d_n(P_n) = M_{n-1} = \ker(d_{n-1})$, the chain complex $P_\bullet$ is a resolution of M. (See Figure 2.1.) $\square$

Exercise 2.2.3 Show that if $P_\bullet$ is a complex of projectives with $P_i = 0$ for $i < 0$, then a map $\varepsilon : P_0 \to M$ giving a resolution for M is the same thing as a quasi-isomorphism $\varepsilon : P_\bullet \to M$, where M is considered as a complex concentrated in degree zero.

If we have a projective resolution for M, then we have

$$\cdots \to P_2 \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \to 0,$$

exact and with each P_i projective. As the complex is exact, $H_n(P_\bullet) = 0$ for $n \geq 0$, and $H_{-1}(P_\bullet) = {P_0}/{\operatorname{im} d_1} = M$. Equivalently by exercise 1.1.5, that means the following $\varepsilon : P_\bullet \to M$ is a quasi-isomorphism:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d, "\varepsilon"] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Comparison Theorem 2.2.6 *Let $P_\bullet \xrightarrow{\varepsilon} M$ be a projective resolution of M and $f' : M \to N$ a map in $\mathcal{A}$. Then for every resolution $Q_\bullet \xrightarrow{\eta} N$ of N there is a chain map $f : P_\bullet \to Q_\bullet$ lifting f' in the sense that $\eta \circ f_0 = f' \circ \varepsilon$. The chain map f is unique up to chain homotopy equivalence.*

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "\exists"] & P_1 \arrow[r] \arrow[d, "\exists"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "\exists"] & M \arrow[r] \arrow[d, "f'"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r, "\eta"] & N \arrow[r] & 0
\end{tikzcd}$$

Porism 2.27 The proof will make it clear that the hypothesis that $P \to M$ be a projective resolution is too strong. It suffices to be given a chain complex

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

with the P_i projective. Then for every resolution $Q \to N$ of N, every map $M \to N$ lifts to a map $P \to Q$, which is unique up to chain homotopy. This stronger version of the Comparison Theorem will be used in section 2.7 to construct the external product for Tor.

Proof. We will construct the f_n and show their uniqueness by induction on n, thinking of f_{-1} as f'. Inductively, suppose f_i has been constructed for $i \leq n$ so that $f_{i-1}d = df_i$. In order to construct f_{n+1} we consider the n-cycles of P and Q. If $n = -1$, we set $Z_{-1}(P) = M$ and $Z_{-1}(Q) = N$; if $n \geq 0$, the fact that $f_{n-1}d = df_n$ means that f_n induces a map f'_n from $Z_n(P)$ to $Z_n(Q)$. Therefore we have two diagrams with exact rows

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & P_{n+1} \arrow[r, "d"] \arrow[d, "\exists"] & Z_n(P) \arrow[r] \arrow[d, "f'_n"] & 0 \\
\cdots \arrow[r] & Q_{n+1} \arrow[r, "d"] & Z_n(Q) \arrow[r] & 0
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
0 \arrow[r] & Z_n(P) \arrow[r] \arrow[d, "f'_n"] & P_n \arrow[r] \arrow[d, "f_n"] & P_{n-1} \arrow[d, "f_{n-1}"] \\
0 \arrow[r] & Z_n(Q) \arrow[r] & Q_n \arrow[r] & Q_{n-1}
\end{tikzcd}$$

The universal lifting property of the projective P_{n+1} yields a map f_{n+1} from P_{n+1} to Q_{n+1}, so that $df_{n+1} = f'_n d = f_n d$. This finishes the inductive step and proves that the chain map $f : P \to Q$ exists.

To see uniqueness of f up to chain homotopy, suppose that $g : P \to Q$ is another lift of f' and set $h = f - g$; we will construct a chain contraction $\{s_n : P_n \to Q_{n+1}\}$ of h by induction on n. If $n < 0$, then $P_n = 0$, so we set $s_n = 0$. If $n = 0$, note that since $\eta h_0 = \varepsilon(f' - f') = 0$, the map h_0 sends P_0 to $Z_0(Q) = d(Q_1)$. We use the lifting property of P_0 to get a map $s_0 : P_0 \to Q_1$ so that $h_0 = ds_0 = ds_0 + s_{-1}d$. Inductively, we suppose given maps s_i $(i < n)$ so that $ds_{n-1} = h_{n-1} - s_{n-2}d$ and consider the map $h_n - s_{n-1}d$ from P_n to Q_n. We compute that

$$d(h_n - s_{n-1}d) = dh_n - (h_{n-1} - s_{n-2}d)d = (dh - hd) + s_{n-2}dd = 0.$$

Therefore $h_n - s_{n-1}d$ lands in $Z_n(Q)$, a quotient of Q_{n+1}. The lifting property of P_n yields the desired map $s_n : P_n \to Q_{n+1}$ such that $ds_n = h_n - s_{n-1}d$. $\square$

$$\begin{tikzcd}
 & P_n \arrow[dl, "\exists"'] \arrow[d, "h-sd"] & \\
Q_{n+1} \arrow[r, "d"] & Z_n(Q) \arrow[r] & 0
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
P_n \arrow[r, "d"] \arrow[d, "h"] & P_{n-1} \arrow[r, "d"] \arrow[dl, "s"] \arrow[d, "h"] & P_{n-2} \arrow[dl, "s"] \\
Q_n \arrow[r] & Q_{n-1} &
\end{tikzcd}$$

Here is another way to construct projective resolutions. It is called the Horseshoe Lemma because we are required to fill in the horseshoe-shaped diagram.

Horseshoe Lemma 2.2.8 *Suppose given a diagram*

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
\cdots P_2' \arrow[r] & P_1' \arrow[r] & P_0' \arrow[r, "\varepsilon'"] & A' \arrow[r] \arrow[d, "i_A"] & 0 \\
 & & & A \arrow[d, "\pi_A"] & \\
\cdots P_2'' \arrow[r] & P_1'' \arrow[r] & P_0'' \arrow[r, "\varepsilon''"] & A'' \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

where the column is exact and the rows are projective resolutions. Set $P_n = P_n' \oplus P_n''$. Then the P_n assemble to form a projective resolution P of A, and the right-hand column lifts to an exact sequence of complexes

$$0 \to P' \xrightarrow{i} P \xrightarrow{\pi} P'' \to 0,$$

where $i_n : P_n' \to P_n$ and $\pi_n : P_n \to P_n''$ are the natural inclusion and projection, respectively.

Proof. Lift ε'' to a map $P_0'' \to A$; the direct sum of this with the map $i_A\varepsilon' : P_0' \to A$ gives a map $\varepsilon : P_0 \to A$. The diagram (*) below commutes.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker(\varepsilon') \arrow[r] \arrow[d] & P_0' \arrow[r, "\varepsilon'"] \arrow[d] & A' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker(\varepsilon) \arrow[r] \arrow[d] & P_0 \arrow[r, "\varepsilon"] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker(\varepsilon'') \arrow[r] \arrow[d] & P_0'' \arrow[r, "\varepsilon''"] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{*}$$

The right two columns of (*) are short exact sequences. The Snake Lemma 1.3.2 shows that the left column is exact and that $\operatorname{coker}(\varepsilon) = 0$, so that P_0 maps onto A. This finishes the initial step and brings us to the situation

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
\cdots \arrow[r] & P_1' \arrow[r, "d'"] & \ker(\varepsilon') \arrow[r] \arrow[d] & 0 \\
 & & \ker(\varepsilon) \arrow[d] & \\
\cdots \arrow[r] & P_1'' \arrow[r, "d''"] & \ker(\varepsilon'') \arrow[r] \arrow[d] & 0 \\
 & & 0. &
\end{tikzcd}$$

The filling in of the "horseshoe" now proceeds by induction. $\square$

Exercise 2.2.4 Show that there are maps $\lambda_n : P_n'' \to P_{n-1}'$ so that

$$d = \begin{bmatrix} d' & \lambda \\ 0 & d'' \end{bmatrix}, \qquad \text{i.e.,} \qquad d' \begin{bmatrix} p' \\ p'' \end{bmatrix} = \begin{bmatrix} d'(p') + \lambda(p'') \\ d''(p'') \end{bmatrix}.$$

Let's skip for now.

2.3 Injective Resolutions

An object I in an abelian category $\mathcal{A}$ is *injective* if it satisfies the following universal lifting property: Given an injection $f : A \to B$ and a map $\alpha : A \to I$, there exists at least one map $\beta : B \to I$ such that $\alpha = \beta \circ f$.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[dl, "\exists\beta"] \\
 & I &
\end{tikzcd}$$

We say that $\mathcal{A}$ *has enough injectives* if for every object A in $\mathcal{A}$ there is an injection $A \to I$ with I injective. Note that if $\{I_\alpha\}$ is a family of injectives, then the product $\prod I_\alpha$ is also injective. The notion of injective module was invented by R. Baer in 1940, long before projective modules were thought of.

Baer's Criterion 2.3.1 *A right R-module E is injective if and only if for every right ideal J of R, every map $J \to E$ can be extended to a map $R \to E$.*

Proof. The "only if" direction is a special case of the definition of injective. Conversely, suppose given an R-module B, a submodule A and a map $\alpha : A \to E$. Let $\mathcal{E}$ be the poset of all extensions $\alpha' : A' \to E$ of α to an intermediate submodule $A \subseteq A' \subseteq B$; the partial order is that $\alpha' \le \alpha''$ if α'' extends α'. By Zorn's lemma there is a maximal extension $\alpha' : A' \to E$ in $\mathcal{E}$; we have to show that $A' = B$. Suppose there is some $b \in B$ not in A'. The set $J = \{r \in R \mid br \in A'\}$ is a right ideal of R. By assumption, the map $J \xrightarrow{b} A' \xrightarrow{\alpha'} E$ extends to a map $f : R \to E$. Let A'' be the submodule $A' + bR$ of B and define $\alpha'' : A'' \to E$ by

$$\alpha''(a + br) = \alpha'(a) + f(r), \qquad a \in A' \text{ and } r \in R.$$

This is well defined because $\alpha'(br) = f(r)$ for br in $A' \cap bR$, and α'' extends α', contradicting the existence of b. Hence $A' = B$. $\square$

Exercise 2.3.1 Let $R = \mathbf{Z}/m$. Use Baer's criterion to show that R is an injective R-module. Then show that $\mathbf{Z}/d$ is *not* an injective R-module when $d \mid m$ and some prime p divides both d and $\frac{m}{d}$. (The hypothesis ensures that $\mathbf{Z}/m \neq \mathbf{Z}/d \oplus \mathbf{Z}/e$.)

Let J be an ideal of $R = \mathbf{Z}/m$; then $J = \langle k \rangle$ for k dividing m. Let f be a map $J \to R$. Then we claim that $\operatorname{im} f \subseteq J$. To see this, write a for $[a]$, an equivalence class. If $x \in \operatorname{im} f$, then there exists ℓk with $\ell \in R$ such that $f(\ell k) = x$. Since k divides m, $m = ks$ for some s non zero-divisor, so

$$0 = f(\ell m) = f(\ell k s) = s f(\ell k) = sx.$$

So $sx = mt$ for some t, and since $m = ks$,

$$sx = skt,$$

and so $x = tk$. Therefore $x \in J = \mathbf{Z}/k$, and $\operatorname{im} f \subseteq J$. So for $k \in J$, $f(k) = bk$ for some b, and thus for any $x \in J$, $f(x) = bx$. Thus, to extend f to a map $g : R \to R$, take $g(x) = bx$. See that $g|_J = f$ and that g is clearly an R-module homomorphism. By Baer's, R is an injective R-module.

Now, let d divide m and p be a prime that divides d and $\frac{m}{d}$. We show $\mathbf{Z}/d$ is not an injective R-module. We will use the definition; i.e., we will show that given an injection $f : A \to B$ and a map $\alpha : A \to \mathbf{Z}/d$, there does not exist a map $\beta : B \to \mathbf{Z}/d$ such that $\alpha = \beta f$. Let $A = \mathbf{Z}/p$ and let $B = \mathbf{Z}/m$. There is only one injective map $f : A \to B$; it is the map that sends $1 \mapsto \frac{m}{p}$. To see this, note that for an arbitrary φ,

$$\ker \varphi = \left\{ x \in \mathbf{Z}/p \mid \varphi(x) = k \cdot x = m \right\} = \{p\}$$

if and only if $k = \frac{m}{p}$. Choose α to be the unique injective map $\alpha : A \to \mathbf{Z}/d$. It sends $1 \mapsto \frac{d}{p}$, since

$$\ker \psi = \left\{ x \in \mathbf{Z}/p \mid \psi(x) = \ell \cdot x = d \right\} = \{p\}$$

if and only if $\ell = \frac{d}{p}$, but in particular, it is not the zero map. Now we show that there cannot exist a $\beta : B \to {}^{\mathbf{Z}}\!/\!_{d}$ with $\alpha = \beta f$. Let β be any map $B \to {}^{\mathbf{Z}}\!/\!_{d}$. Then

$$\{x \mid x = dy \text{ for some } y\} \subseteq \ker \beta.$$

Since p divides $\frac{m}{d}$, d must divide $\frac{m}{p}$, i.e., $\frac{m}{p} = dj$ for some j. Since

$$\operatorname{im} f = \left\{ a \mid a = \frac{m}{p} b \text{ for some } b \right\},$$

we thus have

$$\operatorname{im} f = \left\{ a \mid a = \frac{m}{p} b \text{ for some } b \right\} = \{a \mid a = djb \text{ for some } jb\} \subseteq \ker \beta.$$

So therefore $\beta f = 0$ for any β, and hence $\beta f \neq \alpha$, and therefore ${}^{\mathbf{Z}}\!/\!_{d}$ is not an injective module.

Corollary 2.3.2 *Suppose that $R = \mathbf{Z}$, or more generally that R is a principal ideal domain. An R-module A is injective iff it is* divisible, *that is, for every $r \neq 0$ in R and every $a \in A$, $a = br$ for some $b \in A$.*

Example 2.3.3 The divisible abelian groups $\mathbf{Q}$ and $\mathbf{Z}_{p^\infty} = {}^{\mathbf{Z}\left[\frac{1}{p}\right]}\!/\!_{\mathbf{Z}}$ are injective ($\mathbf{Z}\left[\frac{1}{p}\right]$ is the group of rational numbers of the form $\frac{a}{p^n}$, $n \geq 1$). Every injective abelian group is a direct sum of these [KapIAB,section 5]. In particular, the injective abelian group ${}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}$ is isomorphic to $\bigoplus \mathbf{Z}_{p^\infty}$.

We will now show that **Ab** has enough injectives. If A is an abelian group, let $I(A)$ be the product of copies of the injective group ${}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}$, indexed by the set $\operatorname{Hom}_{\mathbf{Ab}}\left(A, {}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}\right)$. Then $I(A)$ is injective, being a product of injectives, and there is a canonical map $e_A : A \to I(A)$. This is our desired injection of A into an injective by the following exercise.

Exercise 2.3.2 Show that e_A is an injection. *Hint*: If $a \in A$, find a map $f : a\mathbf{Z} \to {}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}$ with $f(a) \neq 0$ and extend f to a map $f' : A \to {}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}$.

We follow the hint. Let $a \in A$. There is a map $f : a\mathbf{Z} \to {}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}$ with $f(a) \neq 0$. It is defined without loss of generality by taking the generator a and mapping it to $\frac{1}{\operatorname{ord}(a)} + \mathbf{Z}$, if $\operatorname{ord}(a) < \infty$. The group $\left\langle \frac{1}{\operatorname{ord}(a)} \right\rangle \leq {}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}$ is cyclic of order a, so this is a well-defined injective map.

If $\operatorname{ord}(a) = \infty$, then we seek an map $f : \mathbf{Z} \to {}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}$. Take $\mathbf{Z} \to \mathbf{Q}$ and then project to ${}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}$. If the image of $\mathbf{Z}$ in ${}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}$ is trivial, then that means the image of $\mathbf{Z}$ in $\mathbf{Q}$ lies in $\mathbf{Z} \leq \mathbf{Q}$. We may prevent this by composing by the map $\varphi : \mathbf{Q} \to \mathbf{Q}$, $\varphi(x) = \frac{1}{2}x \notin \mathbf{Z}$. Now we have a nontrivial map $f : \mathbf{Z} \to {}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}$.

Since ${}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}$ is injective and $\langle a \rangle = a\mathbf{Z}$ is an ideal (normal subgroup) of A, by Baer's criterion, f

can be extended to $f' : A \to {}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}$. The hint is proven.

See that the hint is enough to complete the proof, because if ${f'}_a : A \to {}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}$ is a map (writing ${f'}_a$ to mean the extension of $f : a\mathbf{Z} \to {}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}$ for a fixed $a \in A$), then

$$e_A : A \to \prod_{{f'}_{a_\alpha} \in \operatorname{Hom}_{\mathbf{Ab}}\left(A, {}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}\right)} {}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}$$

which is $e_A = ({f'}_{a_0}, {f'}_{a_1}, {f'}_{a_2}, ...)$ is injective, because for every nonzero $a \in A$, ${f'}_a(a) \neq 0$, so $a \notin \ker {f'}_a$, so $a \notin \ker e_A$. Thus, $\ker e_A = 0$, and e_A is injective. This completes the proof.

Exercise 2.3.3 Show that an abelian group A is zero iff $\operatorname{Hom}_{\mathbf{Ab}}\left(A, {}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}\right) = 0$.

Certainly, if $A = 0$, then $\operatorname{Hom}_{\mathbf{Ab}}\left(A, {}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}\right) = 0$, since 0 is an initial object.

Conversely, suppose $A \neq 0$; we show $\operatorname{Hom}_{\mathbf{Ab}}\left(A, {}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}\right) \neq 0$. But this is immediate from the construction in the prior exercise, 2.3.2. Take $a \in A$ and construct the map $f : a\mathbf{Z} \to {}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}$ such that $f(a) \neq 0$. Then the extension $f' \in \operatorname{Hom}_{\mathbf{Ab}}\left(A, {}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}\right)$ still has $f'(a) \neq 0$, so $f' \neq 0$, and thus $\operatorname{Hom}_{\mathbf{Ab}}\left(A, {}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}\right) \neq 0$, as desired.

Now it is a fact, easily verified, that if $\mathcal{A}$ is an abelian category, then the opposite category $\mathcal{A}^{op}$ is also abelian. The definition of injective is dual to that of projective, so we immediately can deduce the following results (2.3.4-2.3.7) by arguing in $\mathcal{A}^{op}$.

Lemma 2.3.4 *The following are equivalent for an object I in an abelian category $\mathcal{A}$:*

1. *I is injective in $\mathcal{A}$.*
2. *I is projective in $\mathcal{A}^{op}$.*
3. *The contravariant functor $\operatorname{Hom}_{\mathcal{A}}(-, I)$ is exact, that is, it takes short exact sequences in $\mathcal{A}$ to short exact sequences in **Ab**.*

Definition 2.3.5 Let M be an object of $\mathcal{A}$. A *right resolution* of M is a cochain complex $I^\bullet$ with $I^i = 0$ for $i < 0$ and a map $M \to I^0$ such that the augmented complex

$$0 \to M \to I^0 \xrightarrow{d} I^1 \xrightarrow{d} I^2 \xrightarrow{d} \cdots$$

is exact. This is the same as a cochain map $M \to I^\bullet$, where M is considered as a complex concentrated in degree 0. It is called an *injective resolution* if each I^i is injective.

Lemma 2.3.6 *If the abelian category $\mathcal{A}$ has enough injectives, then every object in $\mathcal{A}$ has an injective resolution.*

Comparison Theorem 2.3.7 *Let $N \to I^\bullet$ be an injective resolution of N and $f' : M \to N$ a map in $\mathcal{A}$. Then for every resolution $M \to E^\bullet$ there is a cochain map $f : E^\bullet \to I^\bullet$ lifting f'. The map f is unique up to cochain homotopy equivalence.*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "f'"] & E^0 \arrow[r] \arrow[d, "\exists"] & E^1 \arrow[r] \arrow[d, "\exists"] & E^2 \arrow[r] \arrow[d, "\exists"] & \cdots \\
0 \arrow[r] & N \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r, "\eta"] & I^2 \arrow[r] & \cdots
\end{tikzcd}$$

Exercise 2.3.4 Show that I is an injective object in the category of chain complexes iff I is a split exact complex of injectives. Then show that if $\mathcal{A}$ has enough injectives, so does the category $\mathbf{Ch}(\mathcal{A})$ of chain complexes over $\mathcal{A}$. *Hint*: $\mathbf{Ch}(\mathcal{A})^{op} \approx \mathbf{Ch}(\mathcal{A}^{op})$.

For the second part, if $\mathcal{A}$ has enough injectives, then $\mathcal{A}^{op}$ has enough projectives, and by exercise 2.2.2, $\mathbf{Ch}\,(\mathcal{A}^{op})$ has enough projectives. By the hint, $\mathbf{Ch}(\mathcal{A}^{op}) \approx \mathbf{Ch}(\mathcal{A})^{op}$ has enough projectives, and so $\mathbf{Ch}(\mathcal{A})$ has enough injectives, as desired.

The first part should follow in a similar way, dualizing exercise 2.2.1 that shows that projectives in $\mathbf{Ch}(\mathcal{A})$ are split exact complexes of projectives in $\mathcal{A}$. Since exercise 2.2.1 gave us trouble, let's show it explicitly instead, to get the extra practice.

Suppose $I_\bullet \in \operatorname{obj}(\mathbf{Ch}(\mathcal{A}))$ is an injective object. Then for any injection $f : A_\bullet \to B_\bullet$ and map $\alpha : A_\bullet \to I_\bullet$, we get

$$\begin{tikzcd}
0 \arrow[r] & A_\bullet \arrow[r, "f"] \arrow[d, "\alpha"'] & B_\bullet \arrow[dl, dashed] \\
 & I_\bullet &
\end{tikzcd}$$

To see that I_n in $I_\bullet$ is injective in $\mathcal{A}$ for all n, take $A_\bullet$ to be $\cdots \to 0 \to A_n \to 0 \to \cdots$ and $B_\bullet$ to be $\cdots \to 0 \to B_n \to 0 \to \cdots$. Then

$$\begin{tikzcd}
 & 0 \arrow[rr] \arrow[dr] \arrow[dd] & & 0 \arrow[dr] \arrow[dll, dashed] & & \\
0 \arrow[rr] & & A_n \arrow[rr, "f"] \arrow[dr] \arrow[dd, "\alpha"'] & & B_n \arrow[dr] \arrow[dll, dashed] & \\
 & I_{n+1} \arrow[dr] & & 0 \arrow[rr] \arrow[dd] & & 0 \arrow[dll, dashed] \\
 & & I_n \arrow[dr] & & & \\
 & & & I_{n-1} & &
\end{tikzcd}$$

so I_n is injective in $\mathcal{A}$ for all n. Now we show that I is split exact. We show id_I is nulhomotopic; exercise 1.4.3 then implies I is split exact. Consider

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, "\mathrm{id}"] & {\substack{\mathrm{cone}(I) \\ = I[-1] \oplus I}} \arrow[dl, dashed, "\beta"] \\
 & I &
\end{tikzcd}$$

So by injectiveness of I, we get the map $\beta : I[-1] \oplus I \to I$. Denote $\beta(x, y)$ by $s(x) + \mathrm{id}(y)$, $s : I[-1] \to I$. Now, since β is a chain map, $d^I \beta = \beta d^{\mathrm{cone}(I)}$ and

$$\begin{aligned}
d\beta(x, y) &= d(sx + \mathrm{id}\, y) = dsx + dy \\
\beta d(x, y) &= \beta(-dx, dy - (-1)\,\mathrm{id}\, x) = -sdx + dy + x
\end{aligned}$$

since shifting introduces a -1, and so

$$\begin{aligned}
(ds + sd)(x) &= dsx + sdx \\
&= dsx + dy - dy + sdx - x + x \\
&= dsx + dy - (-sdx + dy + x) + x \\
&= d\beta(x, y) - \beta d(x, y) + x \\
&= x.
\end{aligned}$$

Thus $\mathrm{id}_I = ds + sd$ is nulhomotopic, and I is split exact, as desired.

Conversely, we now assume I is a split exact complex of injectives and show that it is injective in $\mathbf{Ch}(\mathcal{A})$. This direction we didn't have problems with in exercise 2.2.1. Simply ("simply") without loss of generality reduce to the case that $I_\bullet$ is $0 \to I_1 \to I_0 \to 0$, I_i injective, and $I_\bullet$ split exact (i.e., $I_1 \cong I_0$), exactly as in 2.2.1. Then, let $f : A_\bullet \to B_\bullet$ be an injection and $\alpha : A_\bullet \to I_\bullet$ a map. We construct $\beta : B_\bullet \to I_\bullet$. Since I is zero outside of degrees 0 and 1, so is β. We get

$$\begin{tikzcd}
0 \arrow[r] & A_0 \arrow[r, "f_0"] \arrow[d, "\alpha_0"] & B_0 \arrow[dl, dotted, "\beta_0"] \\
& I_0 &
\end{tikzcd}$$

by injective-ness of I_0, and let $\beta_1 : B_1 \to I_1$ be

$$B_1 \xrightarrow{d} B_0 \xrightarrow{\beta_0} I_0 \xrightarrow[\sim]{d^{-1}} I_1.$$

To confirm that this works, see that

$$\begin{aligned}
&\beta_0 f_0 = \alpha_0, \text{ and} \\
&\beta_1 f_1 = {d_I}^{-1} \beta_0 d_B f_1 = {d_I}^{-1} \beta_0 f_0 d_A = {d_I}^{-1} \alpha_0 d_A = {d_I}^{-1} d_I \alpha_1 = \alpha_1,
\end{aligned}$$

so $I_\bullet$ is an injective object. Extend this proof to the general case as before; a general injective object $I_\bullet \in \mathrm{obj}(\mathbf{Ch}(\mathcal{A}))$ is $I_\bullet = \bigoplus_{n\in\mathbf{Z}} \big(0 \to \mathrm{im}(d_n) \to \ker(d_{n+1}) \to 0\big)$ and the map is $\sum_{n\in\mathbf{Z}} \beta_n$, when degrees are lined up correctly.

We now show that there are enough injective R-modules for every ring R. Recall that if A is an abelian group and B is a left R-module, then $\mathrm{Hom}_{\mathbf{Ab}}(B, A)$ is a right R-module via the rule $fr : b \mapsto f(rb)$.

Lemma 2.3.8 *For every right R-module M, the natural map*

$$\tau : \mathrm{Hom}_{\boldsymbol{Ab}}(M, A) \to \mathrm{Hom}_{\boldsymbol{mod}-R}(M, \mathrm{Hom}_{\boldsymbol{Ab}}(R, A))$$

is an isomorphism, where $(\tau f)(m)$ is the map $r \mapsto f(mr)$.

Proof. We define a map μ backwards as follows: If $g : M \to \mathrm{Hom}(R, A)$ is an R-module map, μg is the abelian group map sending m to $g(m)(1)$. Since $\tau(\mu g) = g$ and $\mu\tau(f) = f$ (check this!), τ is an isomorphism. $\square$

Definition 2.3.9 A pair of functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ are *adjoint* if there is a natural bijection for all A in $\mathcal{A}$ and B in $\mathcal{B}$:

$$\tau = \tau_{AB} : \mathrm{Hom}_{\mathcal{B}}(L(A), B) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}}(A, R(B)).$$

Here "natural" means that for all $f : A \to A'$ in $\mathcal{A}$ and $g : B \to B'$ in $\mathcal{B}$ the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{B}}(L(A'), B) \arrow[r, "Lf^*"] \arrow[d, "\tau"] & \mathrm{Hom}_{\mathcal{B}}(L(A), B) \arrow[r, "g_*"] \arrow[d, "\tau"] & \mathrm{Hom}_{\mathcal{B}}(L(A), B') \arrow[d, "\tau"] \\
\mathrm{Hom}_{\mathcal{A}}(A', R(B)) \arrow[r, "f^*"] & \mathrm{Hom}_{\mathcal{A}}(A, R(B)) \arrow[r, "Rg_*"] & \mathrm{Hom}_{\mathcal{A}}(A, R(B')).
\end{tikzcd}$$

We call L the *left adjoint* and R the *right adjoint* of this pair. The above lemma states that the forgetful functor from **mod**-R to **Ab** has $\mathrm{Hom}_{\mathbf{Ab}}(R, -)$ as its right adjoint.

Proposition 2.3.10 *If an additive functor $R : \mathcal{B} \to \mathcal{A}$ is right adjoint to an exact functor $L : \mathcal{A} \to \mathcal{B}$ and I is an injective object of $\mathcal{B}$, then $R(I)$ is an injective object of $\mathcal{A}$. (We say that R* preserves injectives*.)*

Dually, if an additive functor $L : \mathcal{A} \to \mathcal{B}$ is left adjoint to an exact functor $R : \mathcal{B} \to \mathcal{A}$ and P is a projective object of $\mathcal{A}$, then $L(P)$ is a projective object of $\mathcal{B}$. (We say that L preserves projectives*.)*

Proof. We must show that $\mathrm{Hom}_{\mathcal{A}}(-, R(I))$ is exact. Given an injection $f : A \to A'$ in $\mathcal{A}$ the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{B}}(L(A'), I) \arrow[r, "Lf^*"] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathcal{B}}(L(A), I) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathcal{A}}(A', R(I)) \arrow[r, "f^*"] & \mathrm{Hom}_{\mathcal{A}}(A, R(I))
\end{tikzcd}$$

commutes by naturality of τ. Since L is exact and I is injective, the top map Lf^* is onto. Hence the bottom map f^* is onto, proving that $R(I)$ is an injective object in $\mathcal{A}$. $\square$

Corollary 2.3.11 *If I is an injective abelian group, then $\mathrm{Hom}_{\boldsymbol{Ab}}(R, I)$ is an injective R-module.*

Exercise 2.3.5 If M is an R-module, let $I(M)$ be the product of copies of $I_0 = \mathrm{Hom}_{\mathbf{Ab}}\left(R, \mathbf{Q}/\mathbf{Z}\right)$, indexed by the set $\mathrm{Hom}_R(M, I_0)$. There is a canonical map $e_M : M \to I(M)$; show that e_M is an injection. Being a product of injectives, $I(M)$ is injective, so this will prove that R-**mod** has enough injectives. An important consequence of this is that every R-module has an injective resolution.

Let M be an R-module. Explicitly, the canonical map

$$e_M : M \to I(M) = \prod_{f_\alpha \in \mathrm{Hom}_R(M,I_0)} I_0$$

is $e_M(m) = \big(f_\alpha(m)\big)_\alpha$. Since $I_0 = \mathrm{Hom}_{\mathbf{Ab}}\left(R, {}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}\right)$ and R is, without loss of generality, a nonzero abelian group (the case where R is 0 is trivial), by Exercise 2.3.3, $I_0 \neq 0$.

Let $m \in M$ be nonzero; we show that $m \notin \ker e_M$, and therefore e_M is injective. We claim that there exists some f_α such that $f_\alpha(m) \neq 0$. This suffices, as then $m \notin \ker f_\alpha$ so $m \notin \ker e_M$.

To prove the claim, see that M is a module, hence an abelian group, and therefore has a cyclic subgroup $\langle m \rangle$. Since $I_0 \neq 0$, define a map $\langle m \rangle \to I_0$ by sending m to a nonzero element in I_0. Now, I_0 is injective by Corollary 2.3.11, and $\langle m \rangle$ is an ideal, so by Baer's criterion, we may extend the map $\langle m \rangle \to I_0$ to a map $M \to I_0$. Call this extension $f_{\alpha'}$, and observe that $f_{\alpha'} \in \mathrm{Hom}_R(M, I_0)$. By construction, $f_{\alpha'}(m) \neq 0$. Thus at the α'-th coordinate, $e_M(m) \neq 0$, and e_M is an injection, as desired.

Example 2.3.12 The category Sheaves(X) of abelian group sheaves (1.6.5) on a topological space X has enough injectives. To see this, we need two constructions. The *stalk* of a sheaf $\mathcal{F}$ at a point $x \in X$ is the abelian group $\mathcal{F}_x = \lim_{\to}\{\mathcal{F}(U) \mid x \in U\}$. "Stalk at x" is an exact functor from Sheaves(X) to $\mathbf{Ab}$. If A is any abelian group, the *skyscraper sheaf* x_*A at the point $x \in X$ is defined to be the presheaf

$$(x_*A)(U) = \begin{cases} A & \text{if } x \in U \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 2.3.6 Show that x_*A is a sheaf and that

$$\mathrm{Hom}_{\mathbf{Ab}}(\mathcal{F}_x, A) \cong \mathrm{Hom}_{\mathrm{Sheaves}(X)}(\mathcal{F}, x_*A)$$

for every sheaf $\mathcal{F}$. Use 2.3.10 to conclude that if A_x is an injective abelian group, then $x_*(A_x)$ is an injective object in Sheaves(X) for each x, and that $\prod_{x \in X} x_*(A_x)$ is also injective.

Let X be a topological space. Recall the definition of a sheaf.

Given the category Open(X) of objects $U \subseteq X$ open sets and arrows $U \to V$ exactly when $U \subseteq V$, a presheaf $\mathcal{F}$ is a functor $\mathcal{F} : \mathrm{Open}(X)^{op} \to \mathbf{Ab}$ where $U \mapsto \mathcal{F}(U)$ and $V \to U \mapsto \mathcal{F}(V) \xrightarrow{res} \mathcal{F}(U)$. A sheaf is a presheaf $\mathcal{F}$ such that if $\{U_i\}$ is an open cover and $s_i \in \mathcal{F}(U_i)$ are sections satisfying $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there exists a unique glued section $s \in \mathcal{F}\left(\bigcup_i U_i\right)$ such that $s|_{U_i} = s_i$.

So, we begin by showing that x_*A is a presheaf. We must show it is a functor, that there is an $x_*A(U)$ for all open $U \subseteq X$, and that if $U \subseteq V$, there is a map $x_*A(V) \to x_*A(V)$.

Certainly, the second and third conditions hold. Given U, we can construct $x_*A(U)$ by its definition. If $U \subseteq V$, then we define the map $x_*A(V) \to x_*A(U)$. There are three cases:

1. $x \in U$, so $x \in V$. In this case, $x_*A(V) = A$ and $x_*A(U) = A$, so $A \mapsto A$.
2. $x \notin U$, but $x \in V$. In this case, $x_*A(V) = A$ while $x_*A(U) = 0$, so $A \mapsto 0$.
3. $x \notin U$ and $x \notin V$. In this case, $x_*A(V) = 0$ and $x_*A(U) = 0$, so $0 \mapsto 0$.

This is well-defined.

It just remains to show that x_*A is a functor. We must show that $\mathrm{id}_U \mapsto \mathrm{id}_{x_*A(U)}$ for all open sets U and that given a composition $U \to V \to W$ in $\mathrm{Open}(X)$, x_*A respects the composition from $U \leftarrow V \leftarrow W$ in $\mathrm{Open}(X)^{op}$.

For the first, see that $U \subseteq U$, so id_U is an arrow in $\mathrm{Open}(X)$. Since $x_*A(U)$ is either A or 0 depending on x, the map determined by $x_*A(U)$ either sends A to A or 0 to 0 (i.e., we are not in case 2 above). Thus it is the identity, as desired.

For the second, we have shown as much in the three cases above that demonstrate the existence of the map. The composition is either constantly A, becomes 0 once x is no longer in the nest of sets, or is constantly 0.

Thus we have shown x_*A is a presheaf. To see it is a sheaf, let $\{U_i\}$ be an open cover of X, and let s_i be a collection of sections in $x_*A(U_i)$ that agree on any intersections. There does exist a unique glued section s on $\bigcup_i U_i$. We can again consider three cases:

1. x is not in any U_i. In this case, every $x_*A(U_i)$ is 0, so every $s_i = 0$. Define $s = 0$.
2. x is in exactly one U_i. Call it U_x. In this case, $x_*A(U_i) = 0$ if $U_i \neq U_x$, so $s_i = 0$ if $s_i \neq s_x$. Define $s = \coprod_i s_i = \coprod_{i \neq x} s_i \amalg s_x = \coprod_{i \neq x} 0 \amalg s_x = s_x$.
3. x is in more than one U_i. In this case, build the open cover up courser so that x is in a single U_i. This is permissible because of the requirement that s_i all agree on any intersections of the cover. We have reduced to case 2.

Therefore, x_*A is a sheaf, as desired.

Next, we must show that $\mathrm{Hom}_{\mathbf{Ab}}(\mathcal{F}_x, A) \cong \mathrm{Hom}_{\mathrm{Sheaves}(X)}(\mathcal{F}, x_*A)$ for every sheaf $\mathcal{F}$. We build an explicit isomorphism. We shall define a map

$$\sigma : \mathrm{Hom}_{\mathrm{Sheaves}(X)}(\mathcal{F}, x_*A) \to \mathrm{Hom}_{\mathbf{Ab}}(\mathcal{F}_x, A);$$

i.e., given a map $\mathcal{F} \xrightarrow{f} x_*A$, we must produce a map $\mathcal{F}_x \to A$. To do this, see that since $\mathcal{F}_x = \varinjlim\{\mathcal{F}(U) \mid x \in U\}$, by the universal property of direct limits (i.e., that $\mathcal{F}_x$ with maps $\mathcal{F}(U_i) \to \mathcal{F}_x$ is a universally repelling target), when $x \in U_i \subseteq U_j$, we have

$$\begin{tikzcd}
\mathcal{F}(U_j) \arrow[rr] \arrow[dr] \arrow[ddr, bend right] & & \mathcal{F}(U_i) \arrow[dl] \arrow[ddl, bend left] \\
& \mathcal{F}_x \arrow[d, dashed, "\exists!"] & \\
& A \arrow[d, equal] & \\
& x_*A(U_i) \arrow[d, equal] & \\
& x_*A(U_j) &
\end{tikzcd}$$

We have the map $\mathcal{F}(U_j) \to \mathcal{F}(U_i)$ since $\mathcal{F}$ is a sheaf and $U_i \subseteq U_j$. We have the maps $\mathcal{F}(U_i) \to \mathcal{F}_x$ by construction of direct limit, and we have maps $\mathcal{F}(U_i) \to x_*A(U_i)$ induced by the given sheaf map $\mathcal{F} \xrightarrow{f} x_*A$. Thus by the universal property, there exists a unique map $\mathcal{F}_x \to A$. Let this map be $\sigma(f)$.

We build an inverse for σ. Let

$$\tau : \mathrm{Hom}_{\mathbf{Ab}}(\mathcal{F}_x, A) \to \mathrm{Hom}_{\mathrm{Sheaves}(X)}(\mathcal{F}, x_*A)$$

be defined as follows. Let $\mathcal{F}_x \xrightarrow{g} A$ be a map of abelian groups. To construct a sheaf map $\mathcal{F} \to x_*A$, it is enough to construct it on open sets U_α: $\coprod_\alpha \mathcal{F}(U_\alpha) \to x_*A(U_\alpha)$. If $x \notin U_\alpha$, then $x_*A(U_\alpha) = 0$, and then $\mathcal{F}(U_\alpha) \to x_*A(U_\alpha)$ is the zero map. If $x \in U_\alpha$, then define the map to be

$$\mathcal{F}(U_\alpha) \to \mathcal{F}_x \xrightarrow{g} A = x_*A(U_\alpha),$$

where the map $\mathcal{F}(U_\alpha) \to \mathcal{F}_x$ is the direct limit map.

Now, see that σ and τ are inverses, thus demonstrating the isomorphism. Given a map $\mathcal{F} \xrightarrow{f} x_*A$, $\tau\sigma(f)$ is defined on U_α by

$$\mathcal{F}(U_\alpha) \to \mathcal{F}_x \xrightarrow{\sigma(f)} A = x_*A(U_\alpha)$$

when it is not the zero map. But this is the map f, since we have the diagram

$$\begin{tikzcd}
\mathcal{F}(U_\alpha) \arrow[dr] \arrow[ddr, bend right, "\widehat{f}"'] & \\
& \mathcal{F}_x \arrow[d, "\sigma(f)"] \\
& x_*A(U_\alpha)
\end{tikzcd}$$

denoting $\widehat{f}$ for the map that determines f on $\mathcal{F}(U_\alpha)$.

And given $\mathcal{F}_x \xrightarrow{g} A$, $\sigma\tau(g)$ is $\sigma\left(\mathcal{F}(U_\alpha) \to \mathcal{F}_x \xrightarrow{g} A\right)$. We have the diagram

$$\begin{tikzcd}
\mathcal{F}(U_\alpha) \arrow[rr] \arrow[dr] \arrow[ddr, bend right, "\tau(g)"'] & & \mathcal{F}(U_i) \arrow[dl] \arrow[ddl, bend left] \\
& \mathcal{F}_x \arrow[d, dashed, "\exists!"', "\sigma\tau(g)"] & \\
& A &
\end{tikzcd}$$

Since the map $\sigma\tau(g)$ is unique and also g makes the diagram commute, it must be the case that $\sigma\tau(g) = g$.

Therefore, σ and τ are inverses, as we wished to show, and $\mathrm{Hom}_{\mathrm{Sheaves}(X)}(\mathcal{F}, x_*A) \cong \mathrm{Hom}_{\mathbf{Ab}}(\mathcal{F}_x, A)$.

•••

Assume the isomorphism is shown; then x_*A is right adjoint to "stalk at x," $-_x$. Fix $x \in X$ and let A_x be an injective abelian group. Since $-_x$ is exact, and A_x is injective, by Proposition 2.3.10, x_*A_x is an injective object of Sheaves(X). Since the product of injectives is injective, $\prod_{x\in X} x_*A_x$ is injective, as desired.

Given a fixed sheaf $\mathcal{F}$, choose an injection $\mathcal{F}_x \to I_x$ with I_x injective in $\mathbf{Ab}$ for each $x \in X$. Combining the natural maps $\mathcal{F} \to x_*\mathcal{F}_x$ with $x_*\mathcal{F}_x \to x_*I_x$ yields a map from $\mathcal{F}$ to the injective sheaf $\mathcal{I} = \prod_{x\in X} x_*(I_x)$.

The map $\mathcal{F} \to \mathcal{I}$ is an injection (see [Gode], for example) showing that Sheaves(X) has enough injectives.

Example 2.3.13 Let I be a small category and $\mathcal{A}$ an abelian category. If the product of any set of objects exists in $\mathcal{A}$ ($\mathcal{A}$ is *complete*) and $\mathcal{A}$ has enough injectives, we will show that the functor category $\mathcal{A}^I$ has enough injectives. For each k in I, the k^{th} coordinate $A \mapsto A(k)$ is an exact functor from $\mathcal{A}^I$ to $\mathcal{A}$. Given A in $\mathcal{A}$, define the functor $k_*A : I \to \mathcal{A}$ by sending $i \in I$ to

$$k_*A(i) = \prod_{\mathrm{Hom}_I(i,k)} A.$$

If $\eta : i \to j$ is a map in I, the map $k_*A(i) \to k_*A(j)$ is determined by the index map $\eta^* : \mathrm{Hom}(j,k) \to \mathrm{Hom}(i,k)$. That is, the coordinate $k_*A(i) \to A$ of this map corresponding to $\varphi \in \mathrm{Hom}(j,k)$ is the projection of $k_*A(i)$ onto the factor corresponding to $\eta^*\varphi = \varphi\eta \in \mathrm{Hom}(i,k)$. If $f : A \to B$ is a map in $\mathcal{A}$, there is a corresponding map $k_*A \to k_*B$ defined slotwise. In this way, k_* becomes an additive functor from $\mathcal{A}$ to $\mathcal{A}^I$, assuming that $\mathcal{A}$ has enough products for k_*A to be defined.

Exercise 2.3.7 Assume that $\mathcal{A}$ is complete and has enough injectives. Show that k_* is right adjoint to the k^{th} coordinate functor, so that k_* preserves injectives by 2.3.10. Given $F \in \mathcal{A}^I$, embed each $F(k)$ in an injective object A_k of $\mathcal{A}$, and let $F \to k_*A_k$ be the corresponding adjoint map. Show that the product $E = \prod_{k \in I} k_*A_k$ exists in $\mathcal{A}^I$, that E is an injective object, and that $F \to E$ is an injection. Conclude that $\mathcal{A}^I$ has enough injectives.

Note that $\mathcal{A}^I$ is the functor category, which is comprised of objects which are functors from I to $\mathcal{A}$ and arrows natural transformations between functors.

Fix $k \in I$. Write $F(k)$ for the k^{th} coordinate functor. We must show that

$$\mathrm{Hom}_{\mathcal{A}}(F(k), B) \cong \mathrm{Hom}_{\mathcal{A}^I}(F, k_*B),$$

where F and k_*B are functors from I to $\mathcal{A}$, $F(k)$ and B are objects in $\mathcal{A}$, elements of $\mathrm{Hom}_{\mathcal{A}^I}(F, k_*B)$ are natural transformations from F to k_*B, and elements of $\mathrm{Hom}_{\mathcal{A}}(F(k), B)$ are maps in $\mathcal{A}$ from $F(k)$ to B. We build the isomorphism. Let

$$\sigma : \mathrm{Hom}_{\mathcal{A}^I}(F, k_*B) \to \mathrm{Hom}_{\mathcal{A}}(F(k), B)$$

be the map defined as follows. For an element $\eta \in \mathrm{Hom}_{\mathcal{A}^I}(F, k_*B)$; i.e., a natural transformation $\eta : F \to k_*B = \prod_{\mathrm{Hom}_I(-,k)} B$, let $\sigma(\eta)$ be $\eta(k)$; i.e., $F(k) \to k_*B(k) = \prod_{\mathrm{Hom}_I(k,k)} B = B$, since the identity arrow $k \to k$ in I is unique. This is an arrow in $\mathcal{A}$ from $F(k)$ to B and thus $\sigma(\eta) \in \mathrm{Hom}_{\mathcal{A}}(F(k), B)$.

To show σ is an isomorphism, we construct its inverse. Let

$$\tau : \mathrm{Hom}_{\mathcal{A}}(F(k), B) \to \mathrm{Hom}_{\mathcal{A}^I}(F, k_*B)$$

be defined as follows. For an element $g \in \mathrm{Hom}_{\mathcal{A}}(F(k), B)$; i.e., an arrow $F(k) \xrightarrow{g} B$, define for $i \in I$ the natural transformation $\tau(g) : F \to k_*B = \prod_{\mathrm{Hom}_I(i,k)} B$, given by the maps $F(i) \to F(k) \xrightarrow{g} B$ in the i^{th} coordinate. Since $\tau(g) : F \to k_*B$, $\tau(g) \in \mathrm{Hom}_{\mathcal{A}^I}(F, k_*B)$.

We now show that $\tau\sigma(\eta) = \eta$ and that $\sigma\tau(g) = g$. Indeed, see that

$$\begin{aligned} \tau\sigma(\eta) &= \tau\left(\eta(k)\right) \\ &= F(i) \to F(k) \xrightarrow{\eta(k)} k_*B(k) \end{aligned}$$

in the i^{th} coordinate, so over all $i \in I$, $\tau\sigma(\eta) = F \xrightarrow{\eta} k_*B = \eta$. And

$$\begin{aligned} \sigma\tau(g) &= \sigma\left(F(i) \to F(k) \xrightarrow{g} B\right) \\ &= F(k) \to F(k) \xrightarrow{g} B \\ &= F(k) \xrightarrow{g} B \\ &= g. \end{aligned}$$

Therefore, the isomorphism is shown, and k_* is right adjoint to the k^{th} coordinate functor (which is exact), so k_* preserves injectives by 2.3.10.

• • •

Next, let $F \in \mathcal{A}^I$. Since $\mathcal{A}$ has enough injectives, for all $k \in I$, $F(k) \hookrightarrow A_k$. Let $F \to k_*A_k$ be the corresponding adjoint map. Since $\mathcal{A}$ is complete, $\mathcal{A}^I$ is complete (we show this below), so

$$E = \prod_{k \in I} k_*A_k$$

exists in $\mathcal{A}^I$. E is an injective object because first, k_*A_k is an injective object by Proposition 2.3.10, and second, the product $E = \prod_{k \in I} k_*A_k$ of injective objects must be injective. Finally, $F \to E$ is monic because $F(k) \to A_k$ monic implies $F \to k_*A_k$ is monic by the adjoint isomorphism, and $F \to E$ is therefore monic in the i^{th} coordinate for all i, and thus is monic, as desired. Therefore, $\mathcal{A}^I$ has enough injectives, since a generic object F has a monic map into an injective object E.

Now to show that $\mathcal{A}$ complete implies $\mathcal{A}^I$ is complete. We must show, given $F, G \in \operatorname{obj}(\mathcal{A}^I)$, $F \times G$ exists in $\mathcal{A}^I$. We can define $F \times G$ for each input $i \in \operatorname{obj}(I)$: define $(F \times G)(i) = F(i) \times G(i)$. See that this satisfies the universal property of products: for every object Y in $\mathcal{A}^I$ and maps $f_1 : Y \to F$, $f_2 : Y \to G$, there exists a unique $f : Y \to F \times G$ such that the following commutes:

$$\begin{tikzcd} & Y \arrow[dl, "f_1"'] \arrow[d, dashed, "f"] \arrow[dr, "f_2"] & \\ F & F \times G \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & G. \end{tikzcd}$$

Such an f is just the map defined componentwise by existence of $f(i)$ for all i in the following diagram, since $\mathcal{A}$ is complete and thus products exist:

$$\begin{tikzcd} & Y(i) \arrow[dl, "f_1(i)"'] \arrow[d, dashed, "f(i)"] \arrow[dr, "f_2(i)"] & \\ F(i) & F(i) \times G(i) \arrow[l, "\pi_1(i)"] \arrow[r, "\pi_2(i)"'] & G(i). \end{tikzcd}$$

Thus $F \times G$ exists in $\mathcal{A}^I$. By induction, we get the existence of finite products, and somehow countable and then uncountable products. Transfinite induction? So $\mathcal{A}^I$ is complete, as desired.

Exercise 2.3.8 Use the isomorphism $(\mathcal{A}^I)^{op} \cong (\mathcal{A}^{op})^{(I^{op})}$ to dualize the previous exercise. That is, assuming that $\mathcal{A}$ is cocomplete and has enough projectives, show that $\mathcal{A}^I$ has enough projectives.

Let $\mathcal{A}$ be cocomplete and have enough projectives. Then $\mathcal{A}^{op}$ is cococomplete (i.e., complete) and has enough injectives, so by Exercise 2.3.7, $(\mathcal{A}^{op})^I$ has enough injectives. By the isomorphism given,

$$(\mathcal{A}^{op})^I \cong \left(\mathcal{A}^{I^{op}}\right)^{op}$$

has enough injectives. So $\mathcal{A}^{I^{op}}$ has enough projectives. As I is an arbitrary small category, I^{op} is also a small category. Thus if $\mathcal{A}^{I^{op}}$ has enough projectives, $\mathcal{A}^I$ has enough projectives, as we wished to show.

2.4 Left Derived Functors

Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor between two abelian categories. If $\mathcal{A}$ has enough projectives, we can construct the *left derived functors* L_iF $(i \geq 0)$ of F as follows. If A is an object of $\mathcal{A}$, choose (once and for all) a projective resolution $P \to A$ and define

$$L_iF(A) = H_i(F(P)).$$

Note that since $F(P_1) \to F(P_0) \to F(A) \to 0$ is exact, we always have $L_0F(A) \cong F(A)$. The aim of this section is to show that the L_*F form a universal homological δ-functor.

Lemma 2.4.1 *The objects $L_iF(A)$ of $\mathcal{B}$ are well defined up to natural isomorphism. That is, if $Q \to A$ is a second projective resolution, then there is a canonical isomorphism:*

$$L_iF(A) = H_i(F(P)) \xrightarrow{\cong} H_i(F(Q)).$$

In particular, a different choice of the projective resolutions would yield new functors $\widehat{L}_iF$, which are naturally isomorphic to the functors L_iF.

Proof. By the Comparison Theorem (2.2.6), there is a chain map $f : P \to Q$ lifting the identity map id_A, yielding a map f_* from $H_iF(P)$ to $H_iF(Q)$. Any other such chain map $f' : P \to Q$ is a chain homotopic to f, so $f_* = {f'}_*$. Therefore, the map f_* is canonical. Similarly, there is a chain map $g : Q \to P$ lifting id_A and a map g_*. Since gf and id_P are both chain maps $P \to P$ lifting id_A, we have

$$g_*f_* = (gf)_* = (\mathrm{id}_P)_* = \text{ identity map on } H_iF(P).$$

Similarly, fg and id_Q both lift id_A, so f_*g_* is the identity. This proves that f_* and g_* are isomorphisms. $\square$

Corollary 2.4.2 *If A is projective, then $L_iF(A) = 0$ for $i \neq 0$.*

***F*-Acyclic Objects 2.4.3** An object Q is called *F-acyclic* if $L_iF(Q) = 0$ for all $i \neq 0$, that is, if the higher derived functors of F vanish on Q. Clearly, projectives are F-acyclic for every right exact functor F, but there are others; flat modules are acyclic for tensor products, for example. An *F-acyclic resolution* of A is a left resolution $Q \to A$ for which each Q_i is F-acyclic. We will see later (using dimension shifting, exercise 2.4.3 and 3.2.8) that we can also compute left derived functors from F-acyclic resolutions, that is, that $L_iF(A) \cong H_i(F(Q))$ for any F-acyclic resolution Q of A.

Lemma 2.4.4 *If $f : A' \to A$ is any map in $\mathcal{A}$, there is a natural map $L_iF(f) : L_iF(A') \to L_iF(A)$ for each i.*

Proof. Let $P' \to A'$ and $P \to A$ be the chosen projective resolutions. The comparison theorem yields a lift of f to a chain map $\widetilde{f}$ from P' to P, hence a map $\widetilde{f}_*$ from $H_iF(P')$ to $H_iF(P)$. Any other lift is chain homotopic to $\widetilde{f}$, so the map $\widetilde{f}_*$ is independent of the choice of $\widetilde{f}$. The map $L_iF(f)$ is $\widetilde{f}_*$. $\square$

Exercise 2.4.1 Show that $L_0F(f) = F(f)$ under the identification $L_0F(A) \cong F(A)$.

Let $f : A' \to A$. By the identification and Lemma 2.4.4 above, $L_0F(f) : L_0F(A') \to L_0F(A)$ is $\widetilde{f}_0 : F(A') \to F(A)$, where $\widetilde{f}$ is the chain map gained by applying the Comparison Theorem 2.2.6 to extend $f : A' \to A$, and $\widetilde{f}_*$ is the induced map on homology. We must show that $\widetilde{f}_0 = F(f)$.

Since F is right exact, $H_0(F(A)) = F(A)$ and $H_0(F(A')) = F(A')$, so $H_0(F(f)) = \widetilde{f}_0$:

$H_0(F(A')) \to H_0(F(A))$ is $F(f) : F(A') \to F(A)$, as H_* is a functor. Thus, the following diagram commutes.

$$\begin{array}{ccc} F(A') & \xrightarrow{F(f)} & F(A) \\ \downarrow \wr & & \downarrow \wr \\ H_0(F(A')) & \xrightarrow{\widetilde{f}_0} & H_0(F(A)) \end{array}$$

Therefore, the map $L_0F(f) = \widetilde{f}_0 = H_0(F(f)) = F(f)$, as desired.

Theorem 2.4.5 *Each L_iF is an additive functor from $\mathcal{A}$ to $\mathcal{B}$.*

Proof. The identity map on P lifts the identity map on A, so $L_iF(\mathrm{id}_A)$ is the identity map. Given maps $A' \xrightarrow{f} A \xrightarrow{g} A''$ and chain maps $\widetilde{f}$, $\widetilde{g}$ lifting f and g, the composite $\widetilde{g}\widetilde{f}$ lifts gf. Therefore $g_*f_* = (gf)_*$, proving that L_iF is a functor. If $f_i : A' \to A$ are two maps with lifts $\widetilde{f}_i$, the sum $\widetilde{f}_1 + \widetilde{f}_2$ lifts $f_1 + f_2$. Therefore $f_{1*} + f_{2*} = (f_1 + f_2)_*$, proving that L_iF is additive. □

Exercise 2.4.2 (Preserving derived functors) If $U : \mathcal{B} \to \mathcal{C}$ is an exact functor, show that

$$U(L_iF) \cong L_i(UF).$$

Forgetful functors such as **mod-**$R \to$ **Ab** are often exact, and it is often easier to compute the derived functors of UF due to the absence of cluttering restrictions.

To show two functors are isomorphic, we must produce a natural transformation η between them which is an isomorphism for every map $\eta_A : UL_iF(A) \to L_iUF(A)$, A in $\mathcal{A}$. Thus, we need to show for any given A in $\mathcal{A}$ and chosen projective resolution $P_\bullet \to A$, that

$$UL_iF(A) = U(H_i(F(P))) \cong H_i(U(F(P))) = L_iUF(A).$$

To do this, write $X = F(P)$ with differentials $\{d_n\}$, a complex in $\mathcal{B}$, and we show

$$U(H_i(X)) = U\left({}^{\ker d_n}\!/_{\operatorname{im} d_{n+1}}\right) \cong {}^{\ker U(d_n)}\!/_{\operatorname{im} U(d_{n+1})} = H_i(U(X)).$$

It is clear that it is enough to show that U respects quotients, kernels, and images, for then

$$U\left({}^{\ker d_n}\!/_{\operatorname{im} d_{n+1}}\right) \cong {}^{U(\ker d_n)}\!/_{U(\operatorname{im} d_{n+1})} \cong {}^{\ker U(d_n)}\!/_{\operatorname{im} U(d_{n+1})}.$$

So, observe that U respects quotients, since the short exact sequence

$$0 \to B_n(X) \to Z_n(X) \to H_n(X) \to 0$$

yields a short exact sequence

$$0 \to U(B_n(X)) \to U(Z_n(X)) \to U(H_n(X)) \to 0,$$

and thus $U\left({}^{Z_n(X)}\!\big/_{B_n(X)} \right) = U(H_n(X)) \cong {}^{U(Z_n(X))}\!\big/_{U(B_n(X))}$, as claimed.

Next, U respects kernels. To see this, observe that the short exact sequence

$$0 \to Z_n(X) \to X_n \xrightarrow{d_n} B_{n-1}(X) \to 0$$

yields the short exact sequence

$$0 \to U(Z_n(X)) \to U(X_n) \xrightarrow{U(d_n)} U(B_{n-1}(X)) \to 0.$$

Therefore, $U(\ker d_n) = U(Z_n(X)) \cong \ker(U(d_n))$, as claimed.

Finally, U respects images; this is clear, as we again have the short exact sequence

$$0 \to U(Z_n(X)) \to U(X_n) \xrightarrow{U(d_n)} U(B_{n-1}(X)) \to 0.$$

Thus, $U(\operatorname{im} d_n) = U(B_{n-1}(X)) \cong \operatorname{im}(U(d_n))$, as claimed.

Therefore, the isomorphism on homology is shown, and hence we have a isomorphism $\eta_A : UL_iF(A) \to L_iUF(A)$ for each A in $\mathcal{A}$. It only remains to see that η is a natural transformation; that is, we must show that for objects A and B in $\mathcal{A}$ and map $f : A \to B$, the following diagram commutes:

$$\begin{tikzcd} UL_iF(A) \arrow[r, "UL_iF(f)"] \arrow[d, "\eta_A"'] & UL_iF(B) \arrow[d, "\eta_B"] \\ L_iUF(A) \arrow[r, "L_iUF(f)"'] & L_iUF(B) \end{tikzcd}$$

In other words, we must show that $UL_iF(f) = L_iUF(f)$; that U commutes with L_i just as it does for objects. As before, choose projective resolutions $P_\bullet \to A$ and $Q_\bullet \to B$ so that the Comparison Theorem 2.2.6 yields a chain map $P \to Q$. By Lemma 2.4.4, there is a unique map $L_iUF(f) : L_iUF(A) \to L_iUF(B)$, treating UF as the right exact functor in the statement of that lemma. On the other hand, we can compute $UL_iF(f)$; write $\widetilde{f} : P \to Q$ for the chain map gained by the Comparison Theorem. This induces a map $\widetilde{f}_*$ on homology, so we have, again by Lemma 2.4.4, $\widetilde{f}_* = L_iF(f) : L_iF(A) \to L_iF(B)$. As U is a functor, we then get the map $UL_iF(f) : UL_iF(A) \to UL_iF(B)$. By our work above,

$$UL_iF(A) \cong L_iUF(A) \text{ and } UL_iF(B) \cong L_iUF(B),$$

so $UL_iF(f) : L_iUF(A) \to L_iUF(B)$. But since $L_iUF(f) : L_iUF(A) \to L_iUF(B)$ is unique, we must have $UL_iF(f) = L_iUF(f)$, and the claim is proven. Therefore, $U(L_iF) \cong L_i(UF)$, as we wished to show.

Theorem 2.4.6 *The derived functors L_*F form a homological δ-functor.*

Proof. Given a short exact sequence

$$0 \to A' \to A \to A'' \to 0,$$

choose projective resolutions $P' \to A'$ and $P'' \to A''$. By the Horseshoe Lemma 2.2.8, there is a projective resolution $P \to A$ fitting into a short exact sequence $0 \to P' \to P \to P'' \to 0$ of projective complexes in $\mathcal{A}$. Since the P''_n are projective, each sequence $0 \to P'_n \to P_n \to P''_n \to 0$ is split exact. As F is additive, each sequence

$$0 \to F(P'_n) \to F(P_n) \overset{\leftarrow}{\longrightarrow} F(P''_n) \to 0$$

is split exact in $\mathcal{B}$. Therefore

$$0 \to F(P') \to F(P) \to F(P'') \to 0$$

is a short exact sequence of chain complexes. Writing out the corresponding long exact homology sequence, we get

$$\cdots \xrightarrow{\partial} L_iF(A') \to L_iF(A) \to L_iF(A'') \xrightarrow{\partial} L_{i-1}F(A') \to L_{i-1}F(A) \to L_{i-1}F(A'') \xrightarrow{\partial} \cdots$$

To see the naturality of the ∂_i, assume we are given a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d, "f'"] & A \arrow[r] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & B' \arrow[r, "i_B"'] & B \arrow[r, "\pi_B"'] & B'' \arrow[r] & 0
\end{tikzcd}$$

in $\mathcal{A}$, and projective resolutions of the corners: $\varepsilon' : P' \to A'$, $\varepsilon'' : P'' \to A''$, $\eta' : Q' \to B'$ and $\eta'' : Q'' \to B''$. Use the Horseshoe Lemma 2.2.8 to get projective resolutions $\varepsilon : P \to A$ and $\eta : Q \to B$. Use the Comparison Theorem 2.2.6 to obtain chain maps $F' : P' \to Q'$ and $F'' : P'' \to Q''$ lifting the maps f' and f'', respectively. We shall show that there is also a chain map $F : P \to Q$ lifting f, and giving a commutative diagram of chain complexes with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & P' \arrow[r] \arrow[d, "F'"] & P \arrow[r] \arrow[d, "F"] & P'' \arrow[r] \arrow[d, "F''"] & 0 \\
0 \arrow[r] & Q' \arrow[r] & Q \arrow[r] & Q'' \arrow[r] & 0.
\end{tikzcd}$$

The naturality of the connecting homomorphism in the long exact homology sequence now translates into the naturality of the ∂_i. In order to produce F, we will construct maps (*not* chain maps) $\gamma_n : P''_n \to Q'_n$ such that F_n is

$$F_n = \begin{bmatrix} F'_n & \gamma_n \\ 0 & F''_n \end{bmatrix} : \begin{matrix} P'_n \\ \oplus \\ P''_n \end{matrix} \longrightarrow \begin{matrix} Q'_n \\ \oplus \\ Q''_n \end{matrix}$$

$$F_n(p', p'') = (F'(p') + \gamma(p''), F''(p'')) .$$

Assuming that F is a chain map over f, this choice of F will yield our commutative diagram of chain complexes. In order for F to be a lifting of f, the map $(\eta F_0 - f\varepsilon)$ from $P_0 = P'_0 \oplus P''_0$ to B must vanish. On P'_0 this is no problem, so this just requires that

$$i_B \eta' \gamma_0 = f\lambda_P - \lambda_Q F''_0$$

as maps from P''_0 to B, where λ_P and λ_Q are the restrictions of ε and η to P''_0 and Q''_0, and i_B is the inclusion of B' in B. There is some map $\beta : P''_0 \to B'$ so that $i_B\beta = f\lambda - \lambda F''_0$ because in B'' we have

$$\pi_B(f\lambda - \lambda F''_0) = f''\pi_A\lambda_P - \pi_B\lambda F''_0 = f''\varepsilon'' - \eta'' F''_0 = 0.$$

We may therefore define γ_0 to be any lift of β to Q'_0.

$$\begin{tikzcd} & P''_0 \arrow[dl, "\gamma_0"'] \arrow[d, "\beta"] & \\ Q'_0 \arrow[r, "\eta'"] & B' \arrow[r] & 0 \end{tikzcd}$$

In order for F to be a chain map, we must have

$$\begin{aligned} dF - Fd &= \left[\begin{pmatrix} d' & \lambda \\ 0 & d'' \end{pmatrix}, \begin{pmatrix} F' & \gamma \\ 0 & F'' \end{pmatrix} \right] \\ &= \begin{pmatrix} d'F' - F'd' & (d'\gamma - \gamma d'' + \lambda F'' - F'\lambda') \\ 0 & d''F'' - F''d'' \end{pmatrix} \end{aligned}$$

vanishing. That is, the map $d'\gamma_n : P''_n \to Q'_{n-1}$ must equal

$$g_n = \gamma_{n-1} d'' - \lambda_n F'_n + F''_{n-1}\lambda_n.$$

Inductively, we may suppose γ_i defined for $i < n$, so that g_n exists. A short calculation, using the inductive formula for $d'\gamma_{n-1}$, show that $d'g_n = 0$. As the complex Q' is exact, the map g_n factors through a map $\beta : P''_n \to d(Q'_n)$. We may therefore define γ_n to be any lift of β to Q'_n. This finishes the construction of the chain map F and the proof. □

Exercise 2.4.3 (Dimension shifting) If $0 \to M \to P \to A \to 0$ is exact with P projective (or F-acyclic 2.4.3), show that $L_iF(A) \cong L_{i-1}F(M)$ for $i \geq 2$ and that $L_1F(A)$ is the kernel of $F(M) \to F(P)$. More generally, show that if

$$0 \to M_m \to P_m \to P_{m-1} \to \cdots \to P_0 \to A \to 0$$

is exact with the P_i projective (or F-acyclic), then $L_iF(A) \cong L_{i-m-1}F(M_m)$ for $i \geq m+2$ and $L_{m+1}F(A)$ is the kernel of $F(M_m) \to F(P_m)$. Conclude that if $P \to A$ is an F-acyclic resolution of A, then $L_iF(A) = H_i(F(P))$.
The object M_m, which obviously depends on the choices made, is called the m^{th} *syzygy* of A. The word "syzygy" comes from astronomy, where it was originally used to describe the alignment of the Sun, Earth, and Moon.

Starting with the specific case, let $0 \to M \to P \to A \to 0$ be a short exact sequence, F a right exact functor, and P a projective (or at least F-acyclic) module. Then we get the corresponding long exact sequence

$$
\begin{tikzcd}
 & & \ddots \arrow[dll, snake it] & \\
L_2F(M) \arrow[r] & L_2F(P) \arrow[r] & L_2F(A) \arrow[dll] & \\
L_1F(M) \arrow[r] & L_1F(P) \arrow[r] & L_1F(A) \arrow[dll] & \\
F(M) \arrow[r] & F(P) \arrow[r] & F(A) \arrow[r] & 0.
\end{tikzcd}
$$

Since P is F-acyclic, $L_iF(P) = 0$ for $i \neq 0$. Then we have

$$
\begin{tikzcd}
 & & \ddots \arrow[dll] & \\
L_2F(M) \arrow[r] & 0 \arrow[r] & L_2F(A) \arrow[dll] & \\
L_1F(M) \arrow[r] & 0 \arrow[r] & L_1F(A) \arrow[dll] & \\
F(M) \arrow[r] & F(P) \arrow[r] & F(A) \arrow[r] & 0,
\end{tikzcd}
$$

and since the sequence is exact, $L_iF(A) \cong L_{i-1}F(M)$ for $i \geq 2$, as desired. By the same long exact sequence, we have

$$0 \to L_1F(A) \to F(M) \to F(P),$$

so $L_1F(A)$ is the kernel of $F(M) \to F(P)$.

Now we move to the more general case. Let

$$0 \to M_m \to P_m \to P_{m-1} \to \cdots \to P_0 \to A \to 0$$

be exact, F a right exact functor, and P_i projective/F-acyclic modules. We may write the long exact sequence above as a sequence of short exact sequences. Denote the maps by

$$0 \to M_m \xrightarrow{g} P_m \xrightarrow{f_m} P_{m-1} \xrightarrow{f_{m-1}} \cdots \xrightarrow{f_1} P_0 \xrightarrow{h} A \to 0;$$

then we have, writing $K_m = \mathrm{im}(f_m) = \ker(f_{m-1})$,

$$
\begin{gathered}
0 \to 0 \to M_m \to \mathrm{im}(g) \to 0 \\
0 \to \mathrm{im}(g) \to P_m \to K_m \to 0 \\
0 \to K_m \to P_{m-1} \to K_{m-1} \to 0 \\
\vdots \\
0 \to K_1 \to P_0 \to \mathrm{im}(h) \to 0 \\
0 \to \mathrm{im}(h) \to A \to 0 \to 0.
\end{gathered}
$$

The first and last short exact sequences are isomorphisms that allow us to simplify thusly:

$$
\begin{gathered}
0 \to M_m \to P_m \to K_m \to 0 \\
0 \to K_m \to P_{m-1} \to K_{m-1} \to 0 \\
\vdots \\
0 \to K_1 \to P_0 \to A \to 0.
\end{gathered}
$$

Then, applying the same argument as above, we see that the corresponding long exact sequences are:

$$
\begin{tikzcd}
 & & \ddots \arrow[dll, bend right] \\
L_2F(M_m) \arrow[r] & L_2F(P_m) \arrow[r] & L_2F(K_m) \arrow[dll, bend right] \\
L_1F(M_m) \arrow[r] & L_1F(P_m) \arrow[r] & L_1F(K_m) \arrow[dll, bend right] \\
F(M_m) \arrow[r] & F(P_m) \arrow[r] & F(K_m) \arrow[r] & 0,
\end{tikzcd}
\quad
\begin{tikzcd}
 & & \ddots \arrow[dll, bend right] \\
L_2F(K_m) \arrow[r] & L_2F(P_{m-1}) \arrow[r] & L_2F(K_{m-1}) \arrow[dll, bend right] \\
L_1F(K_m) \arrow[r] & L_1F(P_{m-1}) \arrow[r] & L_1F(K_{m-1}) \arrow[dll, bend right] \\
F(K_m) \arrow[r] & F(P_{m-1}) \arrow[r] & F(K_{m-1}) \arrow[r] & 0,
\end{tikzcd}
\quad \cdots \quad
\begin{tikzcd}
 & & \ddots \arrow[dll, bend right] \\
L_2F(K_1) \arrow[r] & L_2F(P_0) \arrow[r] & L_2F(A) \arrow[dll, bend right] \\
L_1F(K_1) \arrow[r] & L_1F(P_0) \arrow[r] & L_1F(A) \arrow[dll, bend right] \\
F(K_1) \arrow[r] & F(P_0) \arrow[r] & F(A) \arrow[r] & 0.
\end{tikzcd}
$$

which, by projective/F-acyclic-ness of P_i, is

$$
\begin{tikzcd}
 & & \ddots \arrow[dll, bend right] \\
L_2F(M_m) \arrow[r] & 0 \arrow[r] & L_2F(K_m) \arrow[dll, bend right] \\
L_1F(M_m) \arrow[r] & 0 \arrow[r] & L_1F(K_m) \arrow[dll, bend right] \\
F(M_m) \arrow[r] & F(P_m) \arrow[r] & F(K_m) \arrow[r] & 0,
\end{tikzcd}
\quad
\begin{tikzcd}
 & & \ddots \arrow[dll, bend right] \\
L_2F(K_m) \arrow[r] & 0 \arrow[r] & L_2F(K_{m-1}) \arrow[dll, bend right] \\
L_1F(K_m) \arrow[r] & 0 \arrow[r] & L_1F(K_{m-1}) \arrow[dll, bend right] \\
F(K_m) \arrow[r] & F(P_{m-1}) \arrow[r] & F(K_{m-1}) \arrow[r] & 0,
\end{tikzcd}
\quad \cdots \quad
\begin{tikzcd}
 & & \ddots \arrow[dll, bend right] \\
L_2F(K_1) \arrow[r] & 0 \arrow[r] & L_2F(A) \arrow[dll, bend right] \\
L_1F(K_1) \arrow[r] & 0 \arrow[r] & L_1F(A) \arrow[dll, bend right] \\
F(K_1) \arrow[r] & F(P_0) \arrow[r] & F(A) \arrow[r] & 0.
\end{tikzcd}
$$

This yields that $L_{i-m-1}F(M_m) \cong L_{i-m}F(K_m) \cong L_{i-m+1}F(K_{m-1}) \cong \cdots \cong L_iF(A)$ when

$i \geq m+2$, as desired. Further, we have

$$0 \to L_1F(K_m) \to F(M_m) \to F(P_m),$$

so $L_1F(K_m)$ is the kernel of $F(M_m) \to F(P_m)$. By the isomorphism we have shown, $L_1F(K_m) \cong L_2F(K_{m-1}) \cong \cdots \cong L_{m+1}F(A)$ is the kernel, as we wished to show.

We can therefore conclude

Theorem 2.4.7 *Assume that $\mathcal{A}$ has enough projectives. Then for any right exact functor $F : \mathcal{A} \to \mathcal{B}$, the derived functors L_nF form a universal δ-functor.*

Remark This result was first proven in [CE, III.5], but is commonly attributed to [Tohoku], where the term "universal δ-functor" first appeared.

Proof. Suppose that T_* is a homological δ-functor and that $\varphi_0 : T_0 \to F$ is given. We need to show that φ_0 admits a unique extension to a morphism $\varphi : T_* \to L_*F$ of δ-functors. Suppose inductively that $\varphi_i : T_i \to L_iF$ are already defined for $0 \leq i < n$, and that they commute with all the appropriate δ_i's. Given A in $\mathcal{A}$, select an exact sequence $0 \to K \to P \to A \to 0$ with P projective. Since $L_nF(P) = 0$, this yields a commutative diagram with exact rows:

$$\begin{tikzcd}
& T_n(A) \arrow[r, "\delta_n"] & T_{n-1}(K) \arrow[r] \arrow[d, "\varphi_{n-1}"] & T_{n-1}(P) \arrow[d, "\varphi_{n-1}"] \\
0 \arrow[r] & L_nF(A) \arrow[r, "\delta_n"] & L_{n-1}F(K) \arrow[r] & L_{n-1}F(P).
\end{tikzcd}$$

A diagram chase reveals that there exists a *unique* map $\varphi_n(A)$ from $T_n(A)$ to $L_nF(A)$ commuting with the given δ_n's. We need to show that φ_n is a natural transformation commuting with all δ_n's for all short exact sequences.

To see that φ_n is a natural transformation, suppose given $f : A' \to A$ and an exact sequence $0 \to K' \to P' \to A' \to 0$ with P' projective. As P' is projective we can lift f to $g : P' \to P$, which induces a map $h : K' \to K$.

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r] \arrow[d, "h"] & P' \arrow[r] \arrow[d, "g"] & A' \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & K \arrow[r] & P \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

To see that φ_n commutes with f, we note that in the following diagram that each small quadrilateral commutes.

$$\begin{tikzcd}
T_n(A') \arrow[rrr, "T_n(f)"] \arrow[dr, "\delta"] \arrow[ddd, "\varphi_n(A')"'] & & & T_n(A) \arrow[dl, "\delta"'] \arrow[ddd, "\varphi_n(A)"] \\
& T_{n-1}(K') \arrow[r, "T_{n-1}(h)"] \arrow[d, "\varphi_{n-1}"'] & T_{n-1}(K) \arrow[d, "\varphi_{n-1}"] & \\
& L_{n-1}F(K') \arrow[r, "L_{n-1}F(h)"] & L_{n-1}F(K) & \\
L_nF(A') \arrow[ur, "\delta"] \arrow[rrr, "L_nF(f)"] & & & L_nF(A) \arrow[ul, "\delta"']
\end{tikzcd}$$

A chase reveals that

$$\delta \circ L_n F(f) \circ \varphi_n(A') = \delta \circ \varphi_n(A) \circ T_n(f).$$

Because $\delta : L_n F(A) \to L_{n-1} F(K)$ is monic, we can cancel it from the equation to see that the outer square commutes, that is, that φ_n is a natural transformation. Incidentally, this argument (with $A = A'$ and $f = \mathrm{id}_A$) also shows that $\varphi_n(A)$ doesn't depend on the choice of P.

Finally, we need to verify that φ_n commutes with δ_n. Given a short exact sequence $0 \to A' \to A \to A'' \to 0$ and a chosen exact sequence $0 \to K'' \to P'' \to A'' \to 0$ with P'' projective, we can construct maps f and g making the diagram

$$\begin{tikzcd}
0 \arrow[r] & K'' \arrow[r] \arrow[d, "g"] & P'' \arrow[r] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A' \arrow[r] & A \arrow[r] & A'' \arrow[r] & 0
\end{tikzcd}$$

commute. This yields a commutative diagram

$$\begin{tikzcd}
T_n(A'') \arrow[r, "\delta"] \arrow[d, "\varphi_n"'] & T_{n-1}(K'') \arrow[r, "T(g)"] \arrow[d, "\varphi_{n-1}"] & T_{n-1}(A') \arrow[d, "\varphi_{n-1}"] \\
L_n F(A'') \arrow[r, "\delta"] & L_{n-1} F(K'') \arrow[r, "LF(g)"] & L_{n-1} F(A').
\end{tikzcd}$$

Since the horizontal composites are the δ_n maps of the bottom row, this implies the desired commutativity relation. ☐

Exercise 2.4.4 Show that homology $H_* : \mathbf{Ch}_{\geq 0}(\mathcal{A}) \to \mathcal{A}$ and cohomology $H^* : \mathbf{Ch}^{\geq 0}(\mathcal{A}) \to \mathcal{A}$ are universal δ-functors. *Hint*: Copy the proof above, replacing P by $\sigma_{\geq 0}\,\mathrm{cone}(A)[1]$, where $\mathrm{cone}(A)$ is the mapping cone of exercise 1.5.1. If $\mathcal{A}$ has enough projectives, you may also use the projective objects in $\mathbf{Ch}_{\geq 0}(\mathcal{A})$, which are described in Ex. 2.2.1.

By observation (and Example 2.1.2), homology and cohomology are δ-functors; we only need to show they are universal. That is, we must show that given any other δ-functor T and a natural transformation $\varphi_0 : T_0 \to H_0$, there exists a unique morphism $\varphi : T_* \to H_*$ of δ-functors that extends φ_0. Cohomology is similar. Let's follow the hint and use the structure of Theorem 2.4.7.

Suppose that T_* is a homological δ-functor and that $\varphi_0 : T_0 \to H_0$ is given. We need to show that φ_0 admits a unique extension to a morphism $\varphi : T_* \to H_*$ of δ-functors. Suppose inductively that $\varphi_i : T_i \to H_i$ are already defined for $0 \leq i < n$, and that they commute with all the appropriate δ_is. Given $A_\bullet \in \mathbf{Ch}_{\geq 0}(\mathcal{A})$, select an exact sequence

$$0 \to K_\bullet \to \sigma_{\geq 0}\,\mathrm{cone}(A)[+1]_\bullet \to A_\bullet \to 0.$$

Note $H_n(\sigma_{\geq 0}\,\mathrm{cone}(A)[+1]) = 0$ because $\mathrm{id} : A \to A$ is a quasi-isomorphism and thus by Corollary 1.5.4, $\mathrm{cone}(A)$ is exact, so away from the truncation, $\sigma_{\geq 0}\,\mathrm{cone}(A)[+1]$ is exact and thus its homology is zero. Since $H_n(\sigma_{\geq 0}\,\mathrm{cone}(A)[+1]) = 0$, this yields a commutative diagram with exact rows:

$$\begin{tikzcd} & T_n(A) \arrow[r, "\delta_n"] & T_{n-1}(K) \arrow[r] \arrow[d, "\varphi_{n-1}"] & T_{n-1}(\sigma_{\geq} \operatorname{cone}(A)[+1]) \arrow[d, "\varphi_{n-1}"] \\ 0 \arrow[r] & H_n(A) \arrow[r, "\delta_n"] & H_{n-1}(K) \arrow[r] & H_{n-1}(\sigma_{\geq} \operatorname{cone}(A)[+1]). \end{tikzcd}$$

A diagram chase reveals that there exists a unique map $\varphi_n(A)$ from $T_n(A)$ to $H_n(A)$ commuting with the given δ_ns. We need to show that φ_n is a natural transformation commuting with all δ_ns for all short exact sequences.

To see that φ_n is a natural transformation, suppose we are given $f : A' \to A$ and an exact sequence $0 \to K' \to P' \to A' \to 0$ with P' projective. As P' is projective we can lift f to $g : P' \to \sigma_{\geq 0} \operatorname{cone}(A)[+1]$, which induces a map $h : K' \to K$.

$$\begin{tikzcd} 0 \arrow[r] & K' \arrow[r] \arrow[d, "h"] & P' \arrow[r] \arrow[d, "g"] & A' \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & K \arrow[r] & \sigma_{\geq 0} \operatorname{cone}(A)[+1] \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

To see that φ_n commutes with f, we note that in the following diagram that each small quadrilateral commutes.

$$\begin{tikzcd} T_n(A') \arrow[rrr, "T_n(f)"] \arrow[dr, "\delta"] \arrow[ddd, "\varphi_n(A')"'] & & & T_n(A) \arrow[dl, "\delta"'] \arrow[ddd, "\varphi_n(A)"] \\ & T_{n-1}(K') \arrow[r, "T_{n-1}(h)"] \arrow[d, "\varphi_{n-1}"'] & T_{n-1}(K) \arrow[d, "\varphi_{n-1}"] & \\ & H_{n-1}(K') \arrow[r, "H_{n-1}(h)"] & H_{n-1}(K) & \\ H_n(A') \arrow[rrr, "H_n(f)"] \arrow[ur, "\delta"] & & & H_n(A) \arrow[ul, "\delta"'] \end{tikzcd}$$

A chase reveals that $\delta H_n(f)\varphi_n(A') = \delta\varphi_n(A)T_n(f)$. Because $\delta : H_n(A) \to H_{n-1}(K)$ is monic, we can cancel it from the equation to see that the outer square commutes, that is, that φ_n is a natural transformation.

Finally, we need to verify that φ_n commutes with δ_n. Given a short exact sequence $0 \to A' \to A \to A'' \to 0$ and a chosen exact sequence $0 \to K'' \to P'' \to A'' \to 0$ with P'' projective, we can construct maps f and g making the diagram

$$\begin{tikzcd} 0 \arrow[r] & K'' \arrow[r] \arrow[d, "g"] & P'' \arrow[r] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A' \arrow[r] & A \arrow[r] & A'' \arrow[r] & 0 \end{tikzcd}$$

commute. This yields a commutative diagram

$$\begin{tikzcd}
T_n(A'') \arrow[r, "\delta"] \arrow[d, "\varphi_n"'] & T_{n-1}(K'') \arrow[r, "T(g)"] \arrow[d, "\varphi_{n-1}"] & T_{n-1}(A') \arrow[d, "\varphi_{n-1}"] \\
H_n(A'') \arrow[r, "\delta"] & H_{n-1}(K'') \arrow[r, "H_{n-1}(g)"] & H_{n-1}(A').
\end{tikzcd}$$

Since the horizontal composites are the δ_n maps of the bottom row, this implies the desired commutativity relation.

Cohomology follows in the same way.

Exercise 2.4.5 ([Tohoku]) An additive functor $F : \mathcal{A} \to \mathcal{B}$ is called *effaceable* if for each object A of $\mathcal{A}$ there is a monomorphism $u : A \to I$ such that $F(u) = 0$. We call F *coeffaceable* if for every A there is a surjection $u : P \to A$ such that $F(u) = 0$. Modify the above proof to show that if T_* is a homological δ-functor such that each T_n is coeffaceable (except T_0), then T_* is universal. Dually, show that if T^* is a cohomological δ-functor such that each T^n is effaceable (except T^0), then T^* is universal.

Again, let's just do the homological case. Let T_* be a coeffaceble homological δ-functor. Let S_* be any homological δ-functor. We must show that a natural transformation $\varphi_0 : S_0 \to T_0$ extends uniquely to $\varphi : S_* \to T_*$.

Suppose inductively that $\varphi_i : S_i \to T_i$ are already defined for $0 \leq i < n$, and that they commute with all the appropriate δ_is. Given $A \in \mathcal{A}$, select an exact sequence

$$0 \to K \to P \to A \to 0.$$

Since T_n is coeffaceable, $T_n(P \to A) = 0$, and thus this yields a commutative diagram with exact rows:

$$\begin{tikzcd}
 & S_n(A) \arrow[r, "\delta_n"] & S_{n-1}(K) \arrow[r] \arrow[d, "\varphi_{n-1}"] & S_{n-1}(P) \arrow[d, "\varphi_{n-1}"] \\
0 \arrow[r] & T_n(A) \arrow[r, "\delta_n"] & T_{n-1}(K) \arrow[r] & T_{n-1}(P).
\end{tikzcd}$$

A diagram chase reveals that there exists a unique map $\varphi_n(A)$ from $S_n(A)$ to $T_n(A)$ commuting with the given δ_ns. We need to show that φ_n is a natural transformation commuting with all δ_ns for all short exact sequences.

To see that φ_n is a natural transformation, suppose we are given $f : A' \to A$ and an exact sequence $0 \to K' \to P' \to A' \to 0$ with P' projective. As P' is projective, we can lift f to $g : P' \to P$, which induces a map $h : K' \to K$.

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r] \arrow[d, "h"] & P' \arrow[r] \arrow[d, "g"] & A' \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & K \arrow[r] & P \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

To see that φ_n commutes with f, we note that in the following diagram that each small quadrilateral commutes.

$$\begin{tikzcd}
S_n(A') \arrow[rrr, "S_n(f)"] \arrow[dr, "\delta"] \arrow[ddd, "\varphi_n(A')"'] & & & S_n(A) \arrow[dl, "\delta"'] \arrow[ddd, "\varphi_n(A)"] \\
& S_{n-1}(K') \arrow[r, "S_{n-1}(h)"] \arrow[d, "\varphi_{n-1}"'] & S_{n-1}(K) \arrow[d, "\varphi_{n-1}"] & \\
& T_{n-1}(K') \arrow[r, "T_{n-1}(h)"] & T_{n-1}(K) & \\
T_n(A') \arrow[rrr, "T_n(f)"] \arrow[ur, "\delta"] & & & T_n(A) \arrow[ul, "\delta"']
\end{tikzcd}$$

A chase reveals that $\delta T_n(f)\varphi_n(A') = \delta\varphi_n(A)S_n(f)$. Because $\delta : T_n(A) \to T_{n-1}(K)$ is monic, we can cancel it from the equation to see that the outer square commutes, that is, that φ_n is a natural transformation.

Finally, we need to verify that φ_n commutes with δ_n. Given a short exact sequence $0 \to A' \to A \to A'' \to 0$ and a chosen exact sequence $0 \to K'' \to P'' \to A'' \to 0$ with P'' projective, we can construct maps f and g making the diagram

$$\begin{tikzcd}
0 \arrow[r] & K'' \arrow[r] \arrow[d, "g"] & P'' \arrow[r] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A' \arrow[r] & A \arrow[r] & A'' \arrow[r] & 0
\end{tikzcd}$$

commute. This yields a commutative diagram

$$\begin{tikzcd}
S_n(A'') \arrow[r, "\delta"] \arrow[d, "\varphi_n"'] & S_{n-1}(K'') \arrow[r, "S(g)"] \arrow[d, "\varphi_{n-1}"] & S_{n-1}(A') \arrow[d, "\varphi_{n-1}"] \\
T_n(A'') \arrow[r, "\delta"] & T_{n-1}(K'') \arrow[r, "T(g)"] & T_{n-1}(A').
\end{tikzcd}$$

Since the horizontal composites are the δ_n maps of the bottom row, this implies the desired commutativity relation.

2.5 Right Derived Functors

2.5.1 Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between two abelian categories. If $\mathcal{A}$ has enough injectives, we can construct the *right derived functors* R^iF $(i \geq 0)$ of F as follows. If A is an object of $\mathcal{A}$, choose an injective resolution $A \to I^\bullet$ and define

$$R^iF(A) = H^i(F(I)).$$

Note that since $0 \to F(A) \to F(I^0) \to F(I^1)$ is exact, we always have $R^0F(A) \cong F(A)$.

Since F also defines a right exact functor $F^{op} : \mathcal{A}^{op} \to \mathcal{B}^{op}$, and $\mathcal{A}^{op}$ has enough projectives, we can construct the left derived functors L_iF^{op} as well. Since $I^\bullet$ becomes a projective resolution of A in $\mathcal{A}^{op}$, we see that

$$R^iF(A) = (L_iF^{op})^{op}(A).$$

Therefore all the results about right exact functors apply to left exact functors. In particular, the objects $R^iF(A)$ are independent of the choice of injective resolutions, R^*F is a universal cohomological δ-functor, and $R^iF(I) = 0$ for $i \neq 0$ whenever I is injective. Calling an object Q *F-acyclic* if $R^iF(Q) = 0$ $(i \neq 0)$, as in 2.4.3, we see that the right derived functors of F can also be computed from F-acyclic resolutions.

Definition 2.5.2 (Ext functors) For each R-module A, the functor $F(B) = \mathrm{Hom}_R(A, B)$ is left exact. Its right derived functors are called the *Ext* groups:

$$\mathrm{Ext}^i_R(A, B) = R^i \mathrm{Hom}_R(A, -)(B).$$

In particular, $\mathrm{Ext}^0(A, B)$ is $\mathrm{Hom}(A, B)$, and injectives are characterized by Ext via the following exercise.

Exercise 2.5.1 Show that the following are equivalent.

1. B is an *injective* R-module.
2. $\mathrm{Hom}_R(-, B)$ is an exact functor.
3. $\mathrm{Ext}^i_R(A, B)$ vanishes for all $i \neq 0$ and all A (B is $\mathrm{Hom}_R(A, -)$-acyclic for all A).
4. $\mathrm{Ext}^1_R(A, B)$ vanishes for all A.

$\star$ First, we prove 1. implies 2. Let B be an injective R-module and let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact sequence. Then we claim that

$$0 \to \mathrm{Hom}_R(N, B) \xrightarrow{g_*} \mathrm{Hom}_R(M, B) \xrightarrow{f_*} \mathrm{Hom}_R(L, B) \to 0$$

is a short exact sequence. Note that $\mathrm{Hom}_R(-, B)$ is contravariant. Note as well that for $\varphi \in \mathrm{Hom}_R(N, B)$ and $\psi \in \mathrm{Hom}_R(M, B)$, we see that $g_*(\varphi) = \varphi \circ g \in \mathrm{Hom}_R(M, B)$ and $f_*(\psi) = \psi \circ f \in \mathrm{Hom}_R(L, B)$. We are told that $\mathrm{Hom}_R(-, B)$ is always left exact, but we show it too.

First, see that g_* is monic. Indeed, $\ker(g_*) = \{\varphi : N \to B \mid g_*(\varphi) = \varphi \circ g : M \to B = 0\}$. Since $0 : M \to B$ is equivalent to $0 \circ g : M \to B$, we have $\varphi \circ g = 0 \circ g$, and as g is epi, $\varphi = 0$, so g_* is monic, as desired.

Next, see that $\mathrm{im}(g_*) = \ker(f_*)$. To see that $\mathrm{im}(g_*) \subseteq \ker(f_*)$, simply note that $(f_* \circ g_*)(\varphi) = \varphi \circ g \circ f = \varphi \circ 0 = 0$. To see that $\ker(f_*) \subseteq \mathrm{im}(g_*)$, let $\psi \in \ker(f_*)$. Then $f_*(\psi) = \psi \circ f = 0$. This means $\mathrm{im}(f) \subseteq \ker(\psi)$, and since $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is exact, $\mathrm{im}(f) = \ker(g)$, so $\ker(g) \subseteq \ker(\psi)$. We need to produce a $\mu : N \to B$ such that $g_*(\mu) = \mu \circ g = \psi : M \to B$. We claim $\mu = \widetilde{\psi} \circ \widetilde{g}^{-1}$, where $\widetilde{\psi}$ is the map $\widetilde{\psi} : {}^{M}\!/_{\ker(g)} \to B$ induced by ψ via $\psi = \widetilde{\psi} \circ \pi$, and $\widetilde{g}$ is the isomorphism $\widetilde{g} : {}^{M}\!/_{\ker(g)} \to N$ induced by g via $g = \widetilde{g} \circ \pi$. Then we may verify that

$$g_*(\mu) = g_*(\widetilde{\psi} \circ \widetilde{g}^{-1}) = \widetilde{\psi} \circ \widetilde{g}^{-1} \circ g = \widetilde{\psi} \circ \widetilde{g}^{-1} \circ \widetilde{g} \circ \pi = \widetilde{\psi} \circ \pi = \psi.$$

Thus $\mathrm{im}(g_*) = \ker(f_*)$.

Finally, see that f_* is epi. Let $\vartheta \in \mathrm{Hom}_R(L, B)$; we must show there exists a $\psi \in \mathrm{Hom}_R(M, B)$ such that $f_*(\psi) = \psi \circ f = \vartheta$. We use the injective-ness of B. See that we have

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "\vartheta"'] & M \arrow[dl, dashed, "\exists\psi"] \\ & B, & \end{tikzcd}$$

as desired. Therefore, $\mathrm{Hom}_R(-, B)$ is exact.

$\star$ Next, we prove 2. implies 1. Suppose $\mathrm{Hom}_R(-, B)$ is exact. We need to show that B is injective; that is, that given an injection $f : X \to Y$ and a map $\alpha : X \to B$, there exists a map $\beta : Y \to B$ such that

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[dl, dashed, "\exists\beta"] \\ & B. & \end{tikzcd}$$

So $0 \to X \xrightarrow{f} Y$ exact implies that $\mathrm{Hom}_R(Y, B) \xrightarrow{f_*} \mathrm{Hom}_R(X, B) \to 0$ is exact by hypothesis. That means f_* is epi, so given $\alpha \in \mathrm{Hom}_R(X, B)$, there exists $\beta \in \mathrm{Hom}_R(Y, B)$ such that $f_*(\beta) = \beta \circ f = \alpha$, and thus B is injective, as desired.

(Note that 1. if and only if 2. is the content of Lemma 2.3.4.)

$\star$ Next, we prove 1. implies 3. Recall that

$$\mathrm{Ext}^i_R(A, B) = R^i \mathrm{Hom}_R(A, -)(B).$$

Mentioned above (the dual of Corollary 2.4.2), since B is injective, $R^i \mathrm{Hom}_R(A, -)(B) = 0$ for $i \geq 0$.

$\star$ Next, 3. implies 4. is trivial. If $\mathrm{Ext}^i_R(A, B) = 0$ for $i \neq 0$, then certainly it is zero for $i = 1$.

$\star$ Finally, we prove 4. implies 1. Suppose $\mathrm{Ext}^1_R(A, B) = 0$ for all A. We need to show that B is injective. Let $0 \to B \to I^0 \to I^1 \to \cdots$ be an injective resolution. It follows that

$$0 \to B \xrightarrow{\varphi} I^0 \xrightarrow{\psi} I^0/B \to 0$$

is a short exact sequence. Write $A = I^0/B$, and we therefore get the long exact sequence of the derived functor Ext:

$$0 \to \mathrm{Hom}(A, B) \xrightarrow{\varphi_*} \mathrm{Hom}(A, I^0) \xrightarrow{\psi_*} \mathrm{Hom}(A, A) \xrightarrow{\delta} \mathrm{Ext}(A, B) = 0 \to \cdots$$

Since ψ_* is epi, given the identity $\mathrm{id}_A \in \mathrm{Hom}(A, A)$, there exists $\mu \in \mathrm{Hom}(A, I^0)$ such that $\psi_*(\mu) = \psi\mu = \mathrm{id}_A$, so by **Construction of** $\mathrm{Ext}^1_R(A, B)$,

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, "\varphi"] & I^0 \arrow[r, "\psi"] & A \arrow[l, bend right, "\mu"'] \arrow[r] & 0 \end{tikzcd}$$

is split, and therefore $I^0 \cong B \oplus A$.

Finally, we show that $B \oplus A$ is injective if and only if B and A are injective. Since $I^0 \cong B \oplus A$ is injective, this will complete the proof.

For the forward direction, assume $B \oplus A$ is injective. Given a monomorphism $f : X \to Y$ and a map $\gamma : X \to B \oplus A$, there exists $\alpha : Y \to B \oplus A$ such that

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "\gamma"'] & Y \arrow[dl, dashed, "\exists\alpha"] \\ & B \oplus A & \end{tikzcd}$$

commutes. To see B is injective, see that given a map $\lambda : X \to B$, we can factor λ as $X \xrightarrow{\lambda\oplus 0} B \oplus A \xrightarrow{\pi_B} B$. As $B \oplus A$ is injective, we get $\alpha : Y \to B \oplus A$, which we can then compose with π_B to get a map $Y \to B$.

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "\lambda\oplus 0"'] & Y \arrow[dl, "\alpha"] \\ & B \oplus A \arrow[d, "\pi_B"'] & \\ & B. & \end{tikzcd}$$

Thus, given $0 \to X \xrightarrow{f} Y$ and $X \xrightarrow{\lambda} B$, we get a map $Y \xrightarrow{\pi_B\alpha} B$ such that $\pi_B\alpha f = \lambda$, and B is injective, as desired. The summand A is injective by an identical argument. Therefore, B is injective and the proof is completed, but we continue for the sake of more math.

For the backward direction, assume B and A are injective. Let $f : X \to Y$ be a monomorphism and $\gamma : X \to B \oplus A$ a map. We have commutative diagrams

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "\gamma"'] & Y \arrow[ddl, dashed, "\exists\alpha_B"] \\ & B \oplus A \arrow[d, "\pi_B"'] & \\ & B & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "\gamma"'] & Y \arrow[ddl, dashed, "\exists\alpha_A"] \\ & B \oplus A \arrow[d, "\pi_A"'] & \\ & A. & \end{tikzcd}$$

By universal property of products, since we have $Y \xrightarrow{\alpha_B} B$ and $Y \xrightarrow{\alpha_A} A$, there exists a unique map $\alpha : Y \to B \oplus A$ such that $\alpha_B = \pi_B\alpha$ and $\alpha_A = \pi_A\alpha$. Thus we have $\pi_B\gamma = \alpha_B f = \pi_B\alpha f$, and since π_B is an epimorphism, $\gamma = \alpha f$, and therefore $B \oplus A$ is injective, as we wished to show.

We have therefore shown

$$\begin{tikzcd} 1. \arrow[r, Rightarrow, shift left] \arrow[d, Rightarrow] & 2. \arrow[l, Rightarrow, shift left] \\ 3. \arrow[r, Rightarrow] & 4. \arrow[ul, Rightarrow] \end{tikzcd}$$

The behavior of Ext with respect to the variable A characterizes projectives.

Exercise 2.5.2 Show that the following are equivalent.

1. A is a *projective* R-module.
2. $\mathrm{Hom}_R(A, -)$ is an exact functor.
3. $\mathrm{Ext}^i_R(A, B)$ vanishes for all $i \neq 0$ and all B (A is $\mathrm{Hom}_R(-, B)$-acyclic for all B).
4. $\mathrm{Ext}^1_R(A, B)$ vanishes for all B.

Note that with the assumption mentioned in Example 2.5.3, namely, that 2.7.6 shows that the right derived functors of $\mathrm{Hom}_R(-, B)$ also produce Ext, this exercise just becomes dualizing Exercise 2.5.1. We will proceed without this assumption and prove the equivalence from first principles. We will show

$$\begin{tikzcd} 1. \arrow[r, Rightarrow, shift left] & 2. \arrow[l, Rightarrow, shift left] \arrow[dl, Rightarrow] \\ 3. \arrow[r, Rightarrow] & 4. \arrow[u, Rightarrow] \end{tikzcd}$$

⋆ We begin with 1. implies 2. Let A be projective and $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact sequence. We must show that

$$0 \to \mathrm{Hom}_R(A, L) \xrightarrow{f_*} \mathrm{Hom}_R(A, M) \xrightarrow{g_*} \mathrm{Hom}_R(A, N) \to 0$$

is exact, where if $\varphi \in \mathrm{Hom}_R(A, L)$ and $\psi \in \mathrm{Hom}_R(A, M)$, $f_*(\varphi) = f \circ \varphi$ and $g_*(\psi) = g \circ \psi$.

First, f_* is monic. Indeed, $\ker(f_*) = \{\varphi : A \to L \mid f_*(\varphi) = f \circ \varphi : A \to M = 0\}$. Since $0 : A \to M$ is equivalent to $f \circ 0 : A \to M$, we have $f \circ \varphi = f \circ 0$, and as f is monic, $\varphi = 0$, so f_* is monic, as desired.

Next, see that $\mathrm{im}(f_*) = \ker(g_*)$. To see that $\mathrm{im}(f_*) \subseteq \ker(g_*)$, simply note that $(g_* \circ f_*)(\varphi) = g \circ f \circ \varphi = 0 \circ \varphi = 0$. To see that $\ker(g_*) \subseteq \mathrm{im}(f_*)$, note that exactness of $0 \to L \xrightarrow{f} M \xrightarrow{g} N$ means that $f = \ker(g)$. In categorical terms (see Definition 1.2.1), this means that $gf = 0$ and that if $n : K \to M$ is a map such that $gn = 0$, then there exists a unique map $u : K \to L$ such that $fu = n$.

$$\begin{tikzcd} & K \arrow[dl, dashed, "\exists! u"'] \arrow[d, "n"'] \arrow[dr, "0"] & \\ L \arrow[r, "f"'] & M \arrow[r, "g"'] & N \end{tikzcd}$$

Now, let $\psi : A \to M \in \ker(g_*)$; then $g_*(\psi) = g \circ \psi = 0$. By above, there exists a unique $\mu : A \to L$ such that $\psi = f \circ \mu = f_*(\mu)$, so $\psi \in \mathrm{im}(f_*)$, and $\mathrm{im}(f_*) = \ker(g_*)$, as desired.

Finally, see that g_* is epi. Let $\vartheta \in \mathrm{Hom}_R(A, N)$; we must show there exists a $\psi \in \mathrm{Hom}_R(A, M)$ such that $g_*(\psi) = g \circ \psi = \vartheta$. We use the projective-ness of A. See that we have

$$\begin{tikzcd} & A \arrow[dl, dashed, "\exists \psi"'] \arrow[d, "\vartheta"] & \\ M \arrow[r, "g"'] & N \arrow[r] & 0, \end{tikzcd}$$

as desired. Therefore, $\mathrm{Hom}_R(A, -)$ is exact.

⋆ For 2. implies 1., let $\mathrm{Hom}_R(A, -)$ be exact. We must show A is projective; i.e., given a surjection $g : X \to Y$ and a map $\gamma : A \to Y$, there exists $\beta : A \to X$ such that

$$\begin{tikzcd} & A \arrow[dl, dashed, "\exists\beta"'] \arrow[d, "\gamma"] & \\ X \arrow[r, "g"'] & Y \arrow[r] & 0. \end{tikzcd}$$

Since $X \xrightarrow{g} Y \to 0$ is exact and $\operatorname{Hom}_R(A, -)$ is covariant and exact by assumption, $\operatorname{Hom}_R(A, X) \xrightarrow{g_*} \operatorname{Hom}_R(A, Y) \to 0$ is exact, where $g_*(\varphi)$ with $\varphi : A \to X$ is $g \circ \varphi$. Thus g_* is epi, so let $\gamma \in \operatorname{Hom}_R(A, Y)$, and there exists $\beta \in \operatorname{Hom}_R(A, X)$ such that $g_*(\beta) = g \circ \beta = \gamma$, and therefore A is projective, as desired.

$\star$ Next, we demonstrate 2. implies 3. Let $\operatorname{Hom}_R(A, -)$ be exact. Then if $0 \to L \to M \to N \to 0$ is a short exact sequence, we get that

$$0 \to \operatorname{Hom}_R(A, L) \to \operatorname{Hom}_R(A, M) \to \operatorname{Hom}_R(A, N) \to 0$$

is a short exact sequence. Since $\operatorname{Ext}^i_R(A, B) = R^i \operatorname{Hom}_R(A, -)(B)$, we have the long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}_R(A, L) \arrow[r] & \operatorname{Hom}_R(A, M) \arrow[r] & \operatorname{Hom}_R(A, N) \arrow[dll, out=-30, in=150] \\ & \operatorname{Ext}^1_R(A, L) \arrow[r] & \operatorname{Ext}^1_R(A, M) \arrow[r] & \operatorname{Ext}^1_R(A, N) \arrow[dll, out=-30, in=150] \\ & \operatorname{Ext}^2_R(A, L) \arrow[r] & \operatorname{Ext}^2_R(A, M) \arrow[r] & \operatorname{Ext}^2_R(A, N) \arrow[dll, out=-30, in=150] \\ & \ddots & & \end{tikzcd}$$

By universality of the derived functor and the fact that $\operatorname{Hom}_R(A, -)$ is exact, it must be the case that $\operatorname{Ext}^i_R(A, -) = 0$ for all i.

$\star$ 3. implies 4. is easy; if $\operatorname{Ext}^i_R(A, B) = 0$ for all $i \neq 0$ and all B, then it is zero for $i = 1$.

$\star$ Finally, 4. implies 2. Assume $\operatorname{Ext}^1_R(A, B) = 0$ for all B. We must show $\operatorname{Hom}_R(A, -)$ is exact. Let $0 \to L \to M \to N \to 0$ be a short exact sequence. As Ext is the derived functor of $\operatorname{Hom}_R(A, -)$, we get the long exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_R(A, L) \arrow[r] & \operatorname{Hom}_R(A, M) \arrow[r] & \operatorname{Hom}_R(A, N) \arrow[dll, out=-30, in=150] \\
& \operatorname{Ext}^1_R(A, L) \arrow[r] & \operatorname{Ext}^1_R(A, M) \arrow[r] & \operatorname{Ext}^1_R(A, N) \arrow[dll, out=-30, in=150] \\
& \ddots & &
\end{tikzcd}
$$

Since $\operatorname{Ext}^1_R(A, -) = 0$, we have

$$
\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_R(A, L) \arrow[r] & \operatorname{Hom}_R(A, M) \arrow[r] & \operatorname{Hom}_R(A, N) \arrow[dll, out=-30, in=150] \\
& 0 \arrow[r] & \cdots,
\end{tikzcd}
$$

so $\operatorname{Hom}_R(A, -)$ is exact, as desired.

The notion of derived functor has obvious variations for contravariant functors. For example, let F be a contravariant left exact functor from $\mathcal{A}$ to $\mathcal{B}$. This is the same as a covariant left exact functor from $\mathcal{A}^{op}$ to $\mathcal{B}$, so if $\mathcal{A}$ has enough projectives (i.e., $\mathcal{A}^{op}$ has enough injectives), we can define the right derived functors $R^*F(A)$ to be the cohomology of $F(P_\bullet)$, $P_\bullet \to A$ being a projective resolution in $\mathcal{A}$. This too is a universal δ-functor with $R^0F(A) = F(A)$, and $R^iF(P) = 0$ for $i \neq 0$ whenever P is projective.

Example 2.5.3 For each R-module B, the functor $G(A) = \operatorname{Hom}_R(A, B)$ is contravariant and left exact. It is therefore entitled to right derived functors $R^*G(A)$. However, we will see in 2.7.6 that these are just the functors $\operatorname{Ext}^*(A, B)$. That is,

$$R^* \operatorname{Hom}(-, B)(A) \cong R^* \operatorname{Hom}(A, -)(B) = \operatorname{Ext}^*(A, B).$$

Application 2.5.4 Let X be a topological space. The *global sections* functor Γ from $\operatorname{Sheaves}(X)$ to $\mathbf{Ab}$ is the functor $\Gamma(\mathcal{F}) = \mathcal{F}(X)$. It turns out (see 2.6.1 and exercise 2.6.3 below) that Γ is right adjoint to the constant sheaves functor, so Γ is left exact. The right derived functors of Γ are the *cohomology functors* on X:

$$H^i(X, \mathcal{F}) = R^i\Gamma(\mathcal{F}).$$

The cohomology of a sheaf is arguably the central notion in modern algebraic geometry. For more details about sheaf cohomology, we refer the reader to [Hart].

Exercise 2.5.3 Let X be a topological space and $\{A_x\}$ any family of abelian groups, parametrized by the points $x \in X$. Show that the skyscraper sheaves $x_*(A_x)$ of 2.3.12 as well as their product $\mathcal{F} = \prod x_*(A_x)$ are Γ-acyclic, that is, that $H^i(X, \mathcal{F}) = 0$ for $i \neq 0$. This shows that sheaf cohomology can also be computed from resolutions by products of skyscraper sheaves.

Let $\mathcal{H}$ be a sheaf. In section 2.3, we learned that Sheaves(X) has enough injectives, so we get $0 \to \mathcal{H} \to \mathcal{I}$ for an injective sheaf $\mathcal{I}$. Sinces Sheaves(X) is an abelian category (this is mentioned without proof in section 1.6), the map $\mathcal{H} \to \mathcal{I}$ has a cokernel, so we get the short exact sequence

$$0 \to \mathcal{H} \to \mathcal{I} \to \mathcal{J} \to 0.$$

The derived functor gives rise to the long exact sequence of abelian groups

$$\begin{tikzcd}
0 \arrow[r] & \substack{\Gamma(\mathcal{H}) \\ =\mathcal{H}(X)} \arrow[r] & \substack{\Gamma(\mathcal{I}) \\ =\mathcal{I}(X)} \arrow[r] & \substack{\Gamma(\mathcal{J}) \\ =\mathcal{J}(X)} \arrow[dll, rounded corners] \\
& H^1(X,\mathcal{H}) \arrow[r] & H^1(X,\mathcal{I}) \arrow[r] & H^1(X,\mathcal{J}) \arrow[dll, rounded corners] \\
& H^2(X,\mathcal{H}) \arrow[r] & H^2(X,\mathcal{I}) \arrow[r] & H^2(X,\mathcal{J}) \arrow[dll, rounded corners] \\
& \ddots & &
\end{tikzcd}$$

As $\mathcal{I}$ is an injective object, the right derived functors of it are 0. Thus

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{H}(X) \arrow[r] & \mathcal{I}(X) \arrow[r] & \mathcal{J}(X) \arrow[dll, rounded corners] \\
& H^1(X,\mathcal{H}) \arrow[r] & 0 \arrow[r] & H^1(X,\mathcal{J}) \arrow[dll, rounded corners] \\
& H^2(X,\mathcal{H}) \arrow[r] & 0 \arrow[r] & H^2(X,\mathcal{J}) \arrow[dll, rounded corners] \\
& \ddots, & &
\end{tikzcd}$$

and $H^i(X,\mathcal{H}) \cong H^{i-1}(X,\mathcal{J})$ for $i > 1$.

Next, define a sheaf $\mathcal{F}$ to be *flasque/flabby* if, given $U \subseteq V$, the restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$ is an epimorphism.

We proceed with the proof in steps. The following results, when combined, prove the desired conclusion: that flasque sheaves in general (and $x_*(A_x)$ and $\mathcal{F} = \prod x_*(A_x)$ in specific) are Γ-acyclic.

1. If $\mathcal{H}$ is a flasque sheaf, then $H^1(X, \mathcal{H}) = 0$; i.e., $0 \to \mathcal{H}(X) \to \mathcal{I}(X) \to \mathcal{J}(X) \to 0$ is exact when $0 \to \mathcal{H} \to \mathcal{I} \to \mathcal{J} \to 0$ is a short exact sequence. To see this, let $j \in \mathcal{J}(X)$. We need to show there exists $i \in \mathcal{I}(X)$ such that $i \mapsto j$. Suppose for the sake of contradiction that there is no global section of $\mathcal{I}$ that maps to j. Then there is some open $U \subsetneq X$ with section ι which is maximal with respect to set inclusion that maps to j. Since $U \neq X$, there is another open set $U' \subseteq X$ which does not lie entirely in U and section ι' which maps to j. By the gluing of sheaves, on $U \cap U'$, ι differs from ι' only by an element of $\mathcal{H}(U \cap U')$. But since $U \cap U' \subseteq U'$, the map $\mathcal{H}(U') \to \mathcal{H}(U \cap U')$ is a surjection by hypothesis, so we may lift any section on $U \cap U'$ to a section on U'. Thus ι' agrees with ι on $U \cap U'$, and the gluing axiom extends the section to $U \cup U'$. But we claimed U was maximal, so this contradiction means that $\mathcal{I}(X) \to \mathcal{J}(X)$ is surjective, as desired.

2. If $\mathcal{I}$ is an injective sheaf, then $\mathcal{I}$ is flasque; i.e., if $U \subseteq V$, then $\mathcal{I}(V) \to \mathcal{I}(U) \to 0$. There exists a sheaf $\mathbf{Z}_W$ which is $\mathbf{Z}_W(U) = \mathbf{Z}$ for all U. We can define $\mathbf{Z}_U$ to be

$$\mathbf{Z}_U(W) = \begin{cases} \mathbf{Z}_W(W) & \text{if } W \subseteq U \\ 0 & \text{else.} \end{cases}$$

   By similar construction, define the sheaf $\mathbf{Z}_V$. Let $U \subseteq V$; there is a natural monic map of sheaves $0 \to \mathbf{Z}_U \to \mathbf{Z}_V$. Let $\mathcal{I}$ be injective. Just as in $R - \mathbf{mod}$ in Exercise 2.5.1, $\mathrm{Hom}_{\mathrm{Sheaves}(X)}(-, \mathcal{I})$ is left exact always and right exact when $\mathcal{I}$ is injective. So given $0 \to \mathbf{Z}_U \to \mathbf{Z}_V$, we get

$$\mathrm{Hom}_{\mathrm{Sheaves}(X)}(\mathbf{Z}_V, \mathcal{I}) \to \mathrm{Hom}_{\mathrm{Sheaves}(X)}(\mathbf{Z}_U, \mathcal{I}) \to 0.$$

   One can show that $\mathrm{Hom}_{\mathrm{Sheaves}(X)}(\mathbf{Z}_W, \mathcal{F}) \cong \Gamma(W, \mathcal{F}) = \mathcal{F}(W)$, so

$$\mathcal{I}(V) \to \mathcal{I}(U) \to 0,$$

   as desired.

3. Given $0 \to \mathcal{H} \xrightarrow{\varphi} \mathcal{I} \xrightarrow{\psi} \mathcal{J} \to 0$, if $\mathcal{H}$ and $\mathcal{I}$ are flasque, then $\mathcal{J}$ is flasque. To see this, let $U \subseteq V$. Let $j_U \in \mathcal{J}(U)$; we must show it lifts to $\mathcal{J}(V)$. Since $\mathcal{H}$ is flasque, by part 1., $\mathcal{I}(U) \xrightarrow{\psi_{*,U}} \mathcal{J}(U) \to 0$, so j_U lifts to an element $i_U \in \mathcal{I}(U)$. As $\mathcal{I}$ is flasque, i_U lifts to i_V.

Map i_V to j_V via the map $\mathcal{I}(V) \xrightarrow{\psi_{*,V}} \mathcal{J}(V)$ induced by $\mathcal{I} \xrightarrow{\psi} \mathcal{J}$. Therefore, an element $j_V = (\psi_{*,V})(res)^{-1}(\psi_{*,U})^{-1}(j_U)$ is a lift of j_U, and $\mathcal{J}$ is flasque, as desired.

$$\begin{tikzcd}
i_V \arrow[rr, mapsto] \arrow[ddd, mapsto] & & j_V & \\
& \mathcal{I}(V) \arrow[r, "\psi_{*,V}"] \arrow[d, "res"'] & \mathcal{J}(V) \arrow[d, dotted] & \\
& \mathcal{I}(U) \arrow[r, "\psi_{*,U}"'] \arrow[dd] & \mathcal{J}(U) \arrow[r] & 0. \\
i_U \arrow[rr, mapsto] & & j_U & \\
& 0 & &
\end{tikzcd}$$

Thus, since $H^i(X,\mathcal{H}) \cong H^{i-1}(X,\mathcal{J})$ for $i > 1$, we see that if $\mathcal{H}$ is flasque, then by 3., $\mathcal{J}$ is, and by 1., $H^1(X,\mathcal{H}) = 0$, $H^2(X,\mathcal{H}) \cong H^1(X,\mathcal{J}) = 0$, and inductively,

$$H^i(X,\mathcal{H}) \cong H^{i-1}(X,\mathcal{J}) \cong H^{i-2}(X,\mathcal{J}_1) \cong \cdots \cong H^1(X,\mathcal{J}_{i-2}) = 0$$

for $i \neq 0$. It only remains to show that both $x_*(A_x)$ and $\mathcal{F} = \prod x_*(A_x)$ are flasque.

- The skyscraper sheaf $x_*(A_x)$ is flasque. This is a proof by cases: if $x \in U \subseteq V$, then the map is $A_x \to A_x$ the identity. If $x \notin V \supseteq U$, then the map is $0 \to 0$ the identity. If $x \in V \setminus U$, then the map is $A_x \to 0$ the zero map. All three are surjective.
- The product of flasque sheaves is flasque. Let $\mathcal{G}_i$ be flasque for all $i \in I$. Let $U \subseteq V$ and consider the map

$$\prod_{i \in I} \mathcal{G}_i(V) \to \prod_{i \in I} \mathcal{G}_i(U).$$

Let $\prod g_{i,U} \in \prod_{i \in I} \mathcal{G}_i(U)$. As each $\mathcal{G}_i(V) \to \mathcal{G}_i(U)$ is surjective, for every i, there exists $g_{i,V}$ such that $g_{i,V} \mapsto g_{i,U}$. Then the element $\prod g_{i,V} \mapsto \prod g_{i,U}$, so $\prod_{i \in I} \mathcal{G}_i$ is flasque, as desired.

2.6 Adjoint Functors and Left/Right Exactness

We begin with a useful trick for constructing left and right exact functors.

Theorem 2.6.1 *Let $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ be an adjoint pair of additive functors. That is, there is a natural isomorphism*

$$\tau : \mathrm{Hom}_{\mathcal{B}}(L(A), B) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}}(A, R(B)).$$

Then L is right exact, and R is left exact.

Proof. Suppose that $0 \to B' \to B \to B'' \to 0$ is exact in $\mathcal{B}$. By naturality of τ there is a commutative diagram for every A in $\mathcal{A}$.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{B}}(L(A), B') \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathcal{B}}(L(A), B) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathcal{B}}(L(A), B'') \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(A, R(B')) \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(A, R(B)) \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(A, R(B''))
\end{tikzcd}$$

The top row is exact because $\mathrm{Hom}(LA, -)$ is left exact, so the bottom row is exact for all A. By the Yoneda Lemma 1.6.11,

$$0 \to R(B') \to R(B) \to R(B'')$$

must be exact. This proves that every right adjoint R is left exact. In particular $L^{op} : \mathcal{A}^{op} \to \mathcal{B}^{op}$ (which is a right adjoint) is left exact, that is, L is right exact. $\square$

Remark Left adjoints have left derived functors, and right adjoints have right derived functors. This of course assumes that $\mathcal{A}$ has enough projectives, and that $\mathcal{B}$ has enough injectives for the derived functors to be defined.

Application 2.6.2 Let R be a ring and B a left R-module. The following standard propostion shows that $\otimes_R B$: $\mathbf{mod} - R \to \mathbf{Ab}$ is left adjoint to $\mathrm{Hom}_{\mathbf{Ab}}(B, -)$, so $\otimes_R B$ is right exact. More generally, if S is another ring, and B is an $R - S$ bimodule, then $\otimes_R B$ takes $\mathbf{mod} - R$ to $\mathbf{mod} - S$ and is a left adjoint, so it is right exact.

Proposition 2.6.3 *If B is an $R - S$ bimodule and C a right S-module, then $\mathrm{Hom}_S(B, C)$ is naturally a right R-module by the rule $(fr)(b) = f(rb)$ for $f \in \mathrm{Hom}(B, C)$, $r \in R$ and $b \in B$. The functor $\mathrm{Hom}_S(B, -)$ from $\mathbf{mod} - S$ to $\mathbf{mod} - R$ is right adjoint to $\otimes_R B$. That is, for every R-module A and S-module C there is a natural isomorphism*

$$\tau : \mathrm{Hom}_S(A \otimes_R B, C) \xrightarrow{\cong} \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)).$$

Proof. Given $f : A \otimes_R B \to C$, we define $(\tau f)(a)$ as the map $b \mapsto f(a \otimes b)$ for each $a \in A$. Given $g : A \to \mathrm{Hom}_S(B, C)$, we define $\tau^{-1}(g)$ to be the map defined by the bilinear form $a \otimes b \mapsto g(a)(b)$. We leave the verification that $\tau(f)(a)$ is an S-module map, that $\tau(f)$ is an R-module map, $\tau^{-1}(g)$ is an R-module map, τ is an isomorphism with inverse τ^{-1}, and that τ is natural as an exercise for the reader. $\square$

Definition 2.6.4 Let B be a left R-module, so that $T(A) = A \otimes_R B$ is a right exact functor from $\mathbf{mod} - R$ to $\mathbf{Ab}$. We define the abelian groups

$$\mathrm{Tor}_n^R(A, B) = (L_n T)(A).$$

In particular, $\mathrm{Tor}_0^R(A, B) \cong A \otimes_R B$. Recall that these groups are computed by finding a projective resolution $P \to A$ and taking the homology of $P \otimes_R B$. In particular, if A is a projective R-module, then $\mathrm{Tor}_n(A, B) = 0$ for $n \neq 0$.

More generally, if B is an $R-S$ bimodule, we can think of $T(A) = A \otimes_R B$ as a right exact functor landing in $\mathbf{mod} - S$, so we can think of the $\mathrm{Tor}_n^R(A, B)$ as S-modules. Since the forgetful functor U from $\mathbf{mod} - S$ to $\mathbf{Ab}$ is exact, this generalization does not change the underlying abelian groups, it merely adds an S-module structure, because $U(L_* \otimes B) \cong L_* U(\otimes B)$ as derived functors.

The reader may notice that the functor $A \otimes_R$ is also right exact, so we could also form the derived functors $L_*(A \otimes_R)$. We will see in section 2.7 that this yields nothing new in the sense that $L_*(A \otimes_R)(B) \cong L_*(\otimes_R B)(A)$.

Application 2.6.5 Now we see why the inclusion "incl" of Sheaves(X) into Presheaves(X) is a left exact functor, as claimed in 1.6.7; it is the right adjoint to the sheafification functor. The fact that sheafification is right exact is automatic; it is a theorem that sheafification is exact.

Exercise 2.6.1 Show that the derived functor $R^i(\text{incl})$ sends a sheaf $\mathcal{F}$ to the presheaf $U \mapsto H^i(U, \mathcal{F}|_U)$, where $\mathcal{F}|_U$ is the restriction of $\mathcal{F}$ to U and H^i is the sheaf cohomology of 2.5.4. *Hint*: Compose $R^i(\text{incl})$ with the exact functors Presheaves$(X) \to \mathbf{Ab}$ sending $\mathcal{F}$ to $\mathcal{F}(U)$.

Fix an open set $U \subseteq X$. Following the hint, consider the composition $ER^i(\text{incl})$, where E : Presheaves$(X) \to \mathbf{Ab}$ is the exact functor sending $\mathcal{F} \mapsto \mathcal{F}(U)$. By Exercise 2.4.2, exact functors preserve derived functors, so

$$ER^i(\text{incl}) \cong R^i(E\,\text{incl}).$$

Now, $E\,\text{incl}$: Sheaves$(X) \to \mathbf{Ab}$ sends a sheaf $\mathcal{F}$ to its evaluation $\mathcal{F}(U)$. This is the global sections functor Γ on a subspace $U \subseteq X$. Thus by Application 2.5.4,

$$R^i(E\,\text{incl})(\mathcal{F}) = R^i(\Gamma|_U)(\mathcal{F}) = H^i(U, \mathcal{F}|_U).$$

Now commuting the exact functor E, we see that since E : Presheaves$(X) \to \mathbf{Ab}$ and

$$ER^i(\text{incl})(\mathcal{F}) = H^i(U, \mathcal{F}|_U),$$

$H^i(U, \mathcal{F}|_U)$ is the image of a presheaf $\mathcal{F}'$. Thus

$$R^i(\text{incl})(\mathcal{F}) = \mathcal{F}',$$

where $\mathcal{F}'$ is a presheaf that sends a set U to $H^i(U, \mathcal{F}|_U)$, as desired.

Application 2.6.6 Let $f : X \to Y$ be a continuous map of topological spaces. For any sheaf $\mathcal{F}$ on X, we define the *direct image sheaf* $f_*\mathcal{F}$ on Y by $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}V)$ for every open V in Y. (*Exercise*: Show that $f_*\mathcal{F}$ is a sheaf!) For any sheaf $\mathcal{G}$ on Y, we define the *inverse image sheaf* $f^{-1}\mathcal{G}$ to be the sheafification of the presheaf sending an open set U in X to the direct limit $\varinjlim \mathcal{G}(V)$ over the poset of all open sets V in

Y containing $f(U)$. The following exercise shows that f^{-1} is right exact and that f_* is left exact because they are adjoint. The derived functors $R^i f_*$ are called the *higher direct image sheaf functors* and also play a key role in algebraic geometry. (See [Hart] for more details.)

Exercise 2.6.2 Show that for any sheaf $\mathcal{F}$ on X there is a natural map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$, and that for any sheaf $\mathcal{G}$ on Y there is a natural map $\mathcal{G} \to f_*f^{-1}\mathcal{G}$. Conclude that f^{-1} and f_* are adjoint to each other, that is, that there is a natural isomorphism

$$\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}).$$

Let $f : X \to Y$. Application 2.6.6 asks us to show that $f_*\mathcal{F}$ is a sheaf, so let's do that first.

First, $f_*\mathcal{F}$ is a presheaf. For every open set $V \subseteq Y$, there is an object $f_*\mathcal{F}(V)$, because we have defined it to be $\mathcal{F}(f^{-1}V)$, and as $\mathcal{F}$ is a sheaf, $\mathcal{F}(f^{-1}V)$ exists. Additionally, if $V_1 \subseteq V_2 \subseteq Y$, then $f^{-1}V_1 \subseteq f^{-1}V_2 \subseteq X$, so $\mathcal{F}(f^{-1}V_2) \to \mathcal{F}(f^{-1}V_1)$, and thus we have the restriction map $f_*\mathcal{F}(V_2) \to f_*\mathcal{F}(V_1)$, as desired. That $f_*\mathcal{F}(V) \to f_*\mathcal{F}(V)$ is the identity follows from the fact that it is for $\mathcal{F}(f^{-1}V) \to \mathcal{F}(f^{-1}V)$. That the restrictions respect composition follows from the fact that it does for $\mathcal{F}$.

Second, $f_*\mathcal{F}$ is a sheaf; i.e., it respects the gluing axiom. Let $s_i \in f_*\mathcal{F}(V_i)$, $i \in \{1, 2\}$, such that for V_1 and V_2,

$$s_1|_{V_1 \cap V_2} = s_2|_{V_1 \cap V_2}.$$

Recontextualizing s_i as an element of $\mathcal{F}(f^{-1}V_i)$, we have

$$s_1|_{f^{-1}V_1 \cap f^{-1}V_2} = s_2|_{f^{-1}V_1 \cap f^{-1}V_2}.$$

Thus, as $\mathcal{F}$ is a sheaf, there exists a section $s \in \mathcal{F}(f^{-1}V_1 \cup f^{-1}V_2)$ such that $s|_{f^{-1}V_i} = s_i$ for $i \in \{1, 2\}$. Recontextualizing s as an element of $f_*\mathcal{F}(V_1 \cup V_2)$, we have that $s|_{V_i} = s_i$, and we see that $f_*\mathcal{F}$ respects the gluing axiom and thus is a sheaf, as desired.

•••

We turn to the problem at hand. We must show there is a natural map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$. Let $U \subseteq X$ and define the map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ by

$$f^{-1}f_*\mathcal{F}(U) = \varinjlim_{f(U) \subseteq V \subseteq Y} f_*\mathcal{F}(V) = \varinjlim_{f(U) \subseteq V \subseteq Y} \mathcal{F}(f^{-1}V) \xrightarrow{res_{f^{-1}V, U}} \mathcal{F}(U),$$

since $f(U) \subseteq V$ if and only if $U \subseteq f^{-1}V$. This is a map of presheaves, since we did not sheafify the inverse image sheaf, so to get a map of sheaves, we define sheafification explicitly:

> Let $\mathcal{P}$ be a presheaf. The *sheafification* of $\mathcal{P}$ is a sheaf $\widetilde{\mathcal{P}}$ together with a morphism of presheaves $\eta : \mathcal{P} \to \widetilde{\mathcal{P}}$ such that for any sheaf $\mathcal{Q}$ and morphism of presheaves $\mu : \mathcal{P} \to \mathcal{Q}$, there is a unique morphism of sheaves $\nu : \widetilde{\mathcal{P}} \to \mathcal{Q}$ such that
>
> $$\begin{tikzcd} \mathcal{P} \arrow[rr, "\eta"] \arrow[dr, "\mu"'] & & \widetilde{\mathcal{P}} \arrow[dl, dashed, "\exists! \nu"] \\ & \mathcal{Q} & \end{tikzcd}$$
>
> commutes.

We therefore have

$$\begin{tikzcd} f^{-1}f_*\mathcal{F} \arrow[rr, "\eta"] \arrow[dr, "res_{f^{-1}V,U}"'] & & \widetilde{f^{-1}f_*\mathcal{F}} \arrow[dl, dashed, "\exists"] \\ & \mathcal{F}, & \end{tikzcd}$$

so we have a unique natural map of sheaves, as desired. Next, we show the map $\mathcal{G} \to f_*f^{-1}\mathcal{G}$. Let $V \subseteq Y$, note that $f(f^{-1}V) \subseteq V$ always, and define the map $\mathcal{G} \to f_*f^{-1}\mathcal{G}$ by

$$\mathcal{G}(V) \xrightarrow{res_{V,f(f^{-1}V)}} \varinjlim_{f(f^{-1}V)\subseteq W \subseteq Y} \mathcal{G}(W) = f^{-1}\mathcal{G}(f^{-1}V) = f_*f^{-1}\mathcal{G}(V).$$

This is a map of presheaves, and via sheafification, we get a unique map of sheaves.

To conclude that f^{-1} is left adjoint to f_*, we claim that by above, we have a counit-unit adjunction.

> A *counit-unit adjunction* between two categories is two functors $F : \mathcal{D} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{D}$ with natural transformations $\varepsilon : FG \to \mathrm{id}_{\mathcal{C}}$ and $\eta : \mathrm{id}_{\mathcal{D}} \to GF$ such that
>
> $$F \xrightarrow{\mathrm{id}_F} F = F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F \text{ and } G \xrightarrow{\mathrm{id}_G} G = G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G.$$

Maps $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves on X are defined to be maps $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ of abelian groups for all $U \subseteq X$, such that φ respects the restriction maps; i.e., if $U \subseteq V \subseteq X$, then

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "\varphi_V"] \arrow[d, "res_{V,U}"'] & \mathcal{G}(V) \arrow[d, "res_{V,U}"] \\ \mathcal{F}(U) \arrow[r, "\varphi_U"'] & \mathcal{G}(U) \end{tikzcd}$$

commutes. This is exactly a natural transformation of the functors corresponding to $\mathcal{F}$ and $\mathcal{G}$; namely, we may express a presheaf as a functor $\text{Open}(X)^{op} \to \mathbf{Ab}$. By our work above, $f^{-1} : \text{Sheaves}(Y) \to \text{Sheaves}(X)$ and $f_* : \text{Sheaves}(X) \to \text{Sheaves}(Y)$ are two functors with natural transformations $\varepsilon : f^{-1}f_* \to \text{id}_{\text{Sheaves}(X)}$ and $\eta : \text{id}_{\text{Sheaves}(Y)} \to f_*f^{-1}$ such that

$$f^{-1} \xrightarrow{f^{-1}\eta} f^{-1}f_*f^{-1} \xrightarrow{\varepsilon f^{-1}} f^{-1} \text{ and } f_* \xrightarrow{\eta f_*} f_*f^{-1}f_* \xrightarrow{f_*\varepsilon} f_*$$

are the respective identity transformations of f^{-1} and f_*. It just remains to be seen that counit-unit adjunction implies the adjunction of Homs given.

Lemma If $F : \mathcal{D} \to \mathcal{C}$ is left adjoint to $G : \mathcal{C} \to \mathcal{D}$ via counit-unit adjunction with natural transformations $\varepsilon : FG \to \text{id}_{\mathcal{C}}$ and $\eta : \text{id}_{\mathcal{D}} \to GF$

(i.e., for all $X, Y \in \text{obj}(\mathcal{C})$ and $f \in \text{Hom}_{\mathcal{C}}(X, Y)$, $\varepsilon_Y \circ FG(f) = \text{id}_{\mathcal{C}}(f) \circ \varepsilon_X$, and for all $X, Y \in \text{obj}(\mathcal{D})$ and $g \in \text{Hom}_{\mathcal{D}}(X, Y)$, $\eta_y \circ \text{id}_{\mathcal{D}}(g) = GF(g) \circ \eta_X$)

such that $\text{id}_F = \varepsilon F \circ F\eta$ and $\text{id}_G = G\varepsilon \circ \eta G$, then there is an isomorphism $\text{Hom}_{\mathcal{C}}(FA, B) \cong \text{Hom}_{\mathcal{D}}(A, GB)$.

Proof. Let $f : FA \to B$ and $g : A \to GB$. Define $\Phi(f) = G(f) \circ \eta_A$ and $\Psi(g) = \varepsilon_B \circ F(g)$. Observe the computations:

$$\begin{aligned}
\Psi\Phi(f) = \Psi(G(f) \circ \eta_A) = \varepsilon_B \circ F(G(f) \circ \eta_A) &= \varepsilon_B \circ FG(f) \circ F(\eta_A) \\
&= \text{id}_{\mathcal{C}}(f) \circ \varepsilon_{FA} \circ F(\eta_A) \\
&= f \circ \varepsilon F(A) \circ F\eta(A) \\
&= f \circ \text{id}_{FA} \\
&= f,
\end{aligned}$$

and

$$\begin{aligned}\Phi\Psi(g) = \Phi(\varepsilon_B \circ F(g)) = G(\varepsilon_B \circ F(g)) \circ \eta_A &= G(\varepsilon_B) \circ GF(g) \circ \eta_A \\ &= G(\varepsilon_B) \circ \eta_{GB} \circ \mathrm{id}_{\mathcal{D}}(g) \\ &= G\varepsilon(B) \circ \eta G(B) \circ g \\ &= \mathrm{id}_{GB} \circ g \\ &= g.\end{aligned}$$

Therefore Φ and Ψ are inverses, and $\mathrm{Hom}(FA, B) \cong \mathrm{Hom}(A, GB)$, as desired. $\square$

We can thus conclude that $\mathrm{Hom}_{\mathrm{Sheaves}(X)}(f^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathrm{Sheaves}(Y)}(\mathcal{G}, f_*\mathcal{F})$, as we wished to show.

Exercise 2.6.3 Let $*$ denote the one-point space, so that $\mathrm{Sheaves}(*) \cong \mathbf{Ab}$.

1. If $f : X \to *$ is the collapse map, show that f_* and f^{-1} are the global sections functor Γ and the constant sheaves functor, respectively. This proves that Γ is right adjoint to the constant sheaves functor. By 2.6.1, Γ is left exact, as asserted in 2.5.4.
2. If $x : * \to X$ is the inclusion of a point in X, show that x_* and x^{-1} are the skyscraper sheaf and stalk functors of 2.3.12.

1. Let $f : X \to *$ be the collapse map. For any sheaf $\mathcal{F}$ on X, $f_*\mathcal{F}$ is a sheaf on $*$, computed by $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}V)$ for all $V \subseteq *$ open. Since the only nonempty such V is $*$ itself and $(f_*\mathcal{F})(*) = \mathcal{F}(f^{-1}*) = \mathcal{F}(X) = \Gamma(\mathcal{F})$, we see that f_* is the global sections functor Γ, as required.

   First, note that the constant presheaf with value A is the presheaf that assigns to each nonempty open subset of X the value A, and all of whose restriction maps are the identity map $\mathrm{id}_A : A \to A$. The constant sheaf associated to A is the sheafification of the constant presheaf associated to A. Now, for any sheaf $\mathcal{G}$ on $*$, $f^{-1}\mathcal{G}$ is a sheaf on X, computed by sheafifying the presheaf $\mathcal{P}$ which satisfies

$$\mathcal{P}(U) = \varinjlim_{f(U) \subseteq V \subseteq *} \mathcal{G}(V)$$

for an open set $U \subseteq X$. Since the only open nonempty $V \subseteq *$ is $*$ itself,

$$\mathcal{P}(U) = \mathcal{G}(*) = A$$

for some abelian group A. Thus $\mathcal{P}$ is the presheaf that assigns every $U \subseteq X$ the value A, and thus is the constant presheaf. Its sheafification, $f^{-1}\mathcal{G}$, is thus the constant sheaf, as required.

2. Let $x : * \to X$ be the inclusion of a point in X. For any sheaf $\mathcal{F}$ on $*$, $x_*\mathcal{F}$ is a sheaf on X, computed by $(x_*\mathcal{F})(V) = \mathcal{F}(x^{-1}V)$ for all $V \subseteq X$ open. There are two cases:

   – if $x(*) \in V$, then $* \in x^{-1}V$ so $* = x^{-1}V$, or

   – if $x(*) \notin V$, then $* \notin x^{-1}V$ so $\emptyset = x^{-1}V$.

   In the case that $* = x^{-1}V$, $\mathcal{F}(x^{-1}V) = \mathcal{F}(*) = A$ for some abelian group. In the case that $\emptyset = x^{-1}V$, $\mathcal{F}(x^{-1}V) = \mathcal{F}(\emptyset) = 0$. Thus x_* is the skyscraper sheaf, as required.

   We actually are done now; by Exercise 2.3.6, x_* and stalk at $x -_x$ are adjoint, by Exercise 2.6.2, x_* and x^{-1} are adjoint, and by naturality of adjunction, stalk at x is x^{-1}. Still, we show this explicitly.

   For any sheaf $\mathcal{G}$ on X, $x^{-1}\mathcal{G}$ is a sheaf on $*$, computed by sheafifying the presheaf $\mathcal{P}$ which satisfies

$$\mathcal{P}(U) = \varinjlim_{x(*) \subseteq V \subseteq X} \mathcal{G}(V)$$

   for an open set $U \subseteq *$. The only nonempty such U is $*$ itself, so

$$\mathcal{P}(*) = \varinjlim_{x(*) \in V} \mathcal{G}(V)$$

   Since the stalk of $\mathcal{G}$ at $x = x(*)$ is defined to be

$$\mathcal{G}_x = \varinjlim \{\mathcal{G}(V) \mid x \in V\},$$

   immediately we see that $x^{-1}\mathcal{G}$ is the stalk at x, as required.

Application 2.6.7 (Colimits) Let I be a fixed category. There is a diagonal functor Δ from every category $\mathcal{A}$ to the functor category $\mathcal{A}^I$; if $A \in \mathcal{A}$, then ΔA is the constant functor: $(\Delta A)_i = A$ for all i. Recall that the *colimit* of a functor $F : I \to \mathcal{A}$ is an object of $\mathcal{A}$, written $\operatorname{colim}_{i \in I} F_i$, together with a natural transformation from F to $\Delta(\operatorname{colim} F_i)$, which is universal among natural transformations $F \to \Delta A$ with $A \in \mathcal{A}$. (See the appendix or [MacCW,III.3].) This universal property implies that colim is a functor from $\mathcal{A}^I$ to $\mathcal{A}$, at least when the colimit exists for all $F : I \to \mathcal{A}$.

Exercise 2.6.4 Show that colim is left adjoint to Δ. Conclude that colim is a right exact functor when $\mathcal{A}$ is abelian (and colim exists). Show that pushout (the colimit when I is $\bullet \leftarrow \bullet \to \bullet$) is not an exact functor in **Ab**.

Let us explicitly define $\operatorname{colim} : \mathcal{A}^I \to \mathcal{A}$. Let $F \in \mathcal{A}^I$. If $\alpha : i \to j$ in I and if $f_i : F(i) = F_i \to A$ in $\mathcal{A}$, then $\operatorname*{colim}_{i \in I} F_i \in \mathcal{A}$ is defined to be the object such that the following diagram commutes.

$$\begin{tikzcd}
 & A & \\
 & \operatorname*{colim}_{i\in I} F_i \arrow[u, dashed, "\exists!\gamma"] & \\
F_i \arrow[ur, "\iota_i"] \arrow[uur, bend left, "f_i"] \arrow[rr, "F(\alpha)"'] & & F_j \arrow[ul, "\iota_j"'] \arrow[uul, bend right, "f_j"']
\end{tikzcd}$$

In other words, $\iota_i = \iota_j F(\alpha)$, and if $f_i : F_i \to A$ are any maps for all $i \in I$, then there exists a unique map $\gamma : \operatorname{colim} F_i \to A$ such that $f_i = \gamma \iota_i$ for all $i \in I$.

Let $F : I \to \mathcal{A}$ be a functor, and let B be an object in $\mathcal{A}$. We must show that

$$\operatorname{Hom}_{\mathcal{A}}\left(\operatorname*{colim}_{i\in I} F_i, B\right) \cong \operatorname{Hom}_{\mathcal{A}^I}(F, \Delta B)$$

naturally. Let $f \in \operatorname{Hom}(\operatorname{colim} F_i, B)$. We define the map $\sigma : \operatorname{Hom}(\operatorname{colim} F_i, B) \to \operatorname{Hom}(F, \Delta B)$ by defining $\sigma(f)$ to be the natural transformation $F \to \Delta B$ defined for every $i \in I$ by the map $F_i \xmapsto{\iota_i} \operatorname{colim} F_i \xrightarrow{f} B = \Delta B_i$.

For the other direction, let $\eta \in \operatorname{Hom}(F, \Delta B)$. Define $\tau : \operatorname{Hom}(F, \Delta B) \to \operatorname{Hom}(\operatorname{colim} F_i, B)$ as follows. Since η is a natural transformation, $\eta_i : F_i \to \Delta B_i$ is a map for all $i \in I$, and given $\alpha : i \to j$, the following square commutes:

$$\begin{tikzcd}
\Delta B_i \arrow[r, "\Delta B(\alpha)"] & \Delta B_j \\
F_i \arrow[u, "\eta_i"] \arrow[r, "F(\alpha)"'] & F_j \arrow[u, "\eta_j"']
\end{tikzcd}$$

By definition, $\Delta B_i = B$ for all i. Now, define $\tau(\eta)$ to be the map guaranteed by definition of $\operatorname{colim} F_i$ in the following diagram:

$$\begin{tikzcd}
 & B & \\
 & \operatorname*{colim}_{i\in I} F_i \arrow[u, dashed, "\tau(\eta)"] & \\
F_i \arrow[uur, bend left, "\eta_i"] \arrow[ur, "\iota_i"] \arrow[rr, "F(\alpha)"'] & & F_j \arrow[uul, bend right, "\eta_j"'] \arrow[ul, "\iota_j"']
\end{tikzcd}$$

We claim that σ and τ are inverses, thus demonstrating the isomorphism. To see this, we first compute $\tau\sigma(f)$. Observe that $\sigma(f)$ is $f\iota_i$ for all i, i.e.,

$$\begin{tikzcd}
 & B \\
 & \operatorname*{colim}_{i\in I} F_i \arrow[u, "f"] \\
F_i \arrow[ur, "\iota_i"] &
\end{tikzcd}$$

Then, $\tau(f\iota)$ is the map that exists from the following diagram:

$$\begin{tikzcd}
 & B & \\
 & \operatorname*{colim}_{i\in I} F_i \arrow[u, dashed, "\tau(f\iota)"] & \\
F_i \arrow[uur, bend left, "f\iota_i"] \arrow[ur, "\iota_i"] \arrow[rr, "F(\alpha)"'] & & F_j \arrow[uul, bend right, "f\iota_j"'] \arrow[ul, "\iota_j"']
\end{tikzcd}$$

But $\tau(f\iota)$ is unique and clearly f satsfies such a diagram, so $\tau\sigma(f) = \tau(f\iota) = f$.

For the other direction, we compute $\sigma\tau(\eta)$. $\tau(\eta)$ is the unique map $\operatorname{colim} F_i \to B$ in the picture below.

$$\begin{tikzcd}
 & B & \\
 & \operatorname*{colim}_{i\in I} F_i \arrow[u, dashed, "\tau(\eta)"] & \\
F_i \arrow[uur, bend left, "\eta_i"] \arrow[ur, "\iota_i"] \arrow[rr, "F(\alpha)"'] & & F_j \arrow[uul, bend right, "\eta_j"'] \arrow[ul, "\iota_j"']
\end{tikzcd}$$

Applying σ, we get $\tau(\eta)\iota_i$ for all $i \in I$. By commutivity of the above picture, $\tau(\eta)\iota_i = \eta_i$ for all i, so $\sigma\tau(\eta) = \eta$. Therefore, σ and τ are inverses.

For naturality, let $\varphi : F \to F'$ be a natural transformation and let $\psi : B \to B'$ be a map. We must show that the following diagram commutes:

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{A}}\left(\operatorname*{colim}_{i\in I} F'_i, B\right) \arrow[r, "(\operatorname{colim}\varphi)^*"] \arrow[d, "\sigma"] & \operatorname{Hom}_{\mathcal{A}}\left(\operatorname*{colim}_{i\in I} F_i, B\right) \arrow[r, "\psi_*"] \arrow[d, "\sigma"] & \operatorname{Hom}_{\mathcal{A}}\left(\operatorname*{colim}_{i\in I} F_i, B'\right) \arrow[d, "\sigma"] \\
\operatorname{Hom}_{\mathcal{A}^I}(F', \Delta B) \arrow[r, "\varphi^*"'] & \operatorname{Hom}_{\mathcal{A}^I}(A, \Delta B) \arrow[r, "(\Delta B\psi)_*"'] & \operatorname{Hom}_{\mathcal{A}^I}(A, \Delta B')
\end{tikzcd}$$

For the first square, let $f \in \operatorname{Hom}(\operatorname{colim} F'_i, B)$. We need to show that the natural transformations $\varphi^*\sigma f, \sigma(\operatorname{colim}\varphi)^* f : A \to \Delta B$ are equal; we do so by computing them for all $i \in I$. See that

$$(\varphi^*\sigma f)(i) = \varphi^* f\iota_i = f\iota'_i\varphi, \text{ while}$$
$$(\sigma(\operatorname{colim}\varphi)^* f)(i) = (\operatorname{colim}\varphi)^* f\iota_i = f(\operatorname{colim}\varphi)\iota_i.$$

Now, observe the following diagram, commutative by definition of $\operatorname{colim} F_i$:

$$\begin{tikzcd}
 & \operatorname*{colim}_{i\in I} F'_i & \\
F'_i \arrow[ur, "\iota'_i"] & \operatorname*{colim}_{i\in I} F_i \arrow[u, "\operatorname{colim}\varphi"] & F'_j \arrow[ul, "\iota'_j"'] \\
F_i \arrow[u, "\varphi_i"] \arrow[ur, "\iota_i"] \arrow[rr, "F(\alpha)"'] & & F_j \arrow[u, "\varphi_j"'] \arrow[ul, "\iota_j"']
\end{tikzcd}$$

Using the left parallelogram, $(\operatorname{colim}\varphi)\iota_i = \iota'_i\varphi_i$, so the first square commutes, as desired.

For the second square, let $f \in \operatorname{Hom}(\operatorname{colim} F_i, B)$. We need to show that the natural transformations $(\Delta B\psi)_*\sigma f, \sigma\psi_* f : A \to \Delta B'$ are equal; we do so by computing them for all $i \in I$. See that

$$((\Delta B\psi)_*\sigma f)(i) = (\psi_*\sigma f)(i) = (\psi\sigma f)(i) = \psi f\iota_i, \text{ and}$$
$$(\sigma\psi_* f)(i) = \psi_* f\iota_i = \psi f\iota_i.$$

Thus, the second square commutes, and naturality is shown. Therefore, colim is left adjoint to Δ, as we wished to show.

•••

By Theorem 2.6.1, colim is right exact, as long as colim and Δ are additive. Recall that a functor $F : \mathcal{A} \to \mathcal{B}$ is additive if $\operatorname{Hom}_{\mathcal{A}}(A, A') \to \operatorname{Hom}_{\mathcal{B}}(FA, FA')$ is a group homomorphism; i.e., $F(f + g) = F(f) + F(g)$.

To see that colim : $\mathcal{A}^I \to \mathcal{A}$ is additive, let $f, g : F \to F'$ be arrows in $\mathcal{A}^I$. Then $\operatorname{colim}(f+g)$ is the unique map commuting the following diagram.

$$\begin{tikzcd}
& \operatorname*{colim}_{i \in I} {F'}_i & \\
{F'}_i \arrow[ur, "{\iota'}_i"] & \operatorname*{colim}_{i \in I} F_i \arrow[u, dashed, "\operatorname{colim}(f+g)"] & {F'}_j \arrow[ul, "{\iota'}_j"'] \\
F_i \arrow[u, "(f+g)_i"] \arrow[ur] \arrow[rr] & & F_j \arrow[u, "(f+g)_j"'] \arrow[ul]
\end{tikzcd}$$

Consider the following picture (two diagrams superimposed).

$$\begin{tikzcd}
& \operatorname*{colim}_{i \in I} {F'}_i & \\
{F'}_i \arrow[ur, "{\iota'}_i"] & \operatorname*{colim}_{i \in I} F_i \arrow[u, shift left, "\operatorname{colim}(g)"'] \arrow[u, shift right, "\operatorname{colim}(f)"] & {F'}_j \arrow[ul, "{\iota'}_j"'] \\
F_i \arrow[u, shift right, "f_i"] \arrow[u, shift left, "g_i"'] \arrow[ur] \arrow[rr] & & F_j \arrow[u, shift right, "f_j"] \arrow[u, shift left, "g_j"'] \arrow[ul]
\end{tikzcd}$$

This diagram lies in $\mathcal{A}$, which is an abelian, hence **Ab**-category, so ${\iota'}_i(f_i + g_i) = {\iota'}_i f_i + {\iota'}_i g_i$. Thus we may add all parallel arrows in the diagram above, so we have $\operatorname{colim}(f) + \operatorname{colim}(g)$. And certainly, ${\iota'}_i(f+g)_i = {\iota'}_i(f_i + g_i)$, which means that the above two pictures are identical. Thus, colim is additive.

To see that $\Delta : \mathcal{A} \to \mathcal{A}^I$ is additive, let $f, g : B \to B'$ be arrows in $\mathcal{A}$. Then $\Delta(f+g)$ is the natural transformation $\Delta(f+g)(i) = f + g$ for all $i \in I$, so $\Delta(f+g)(i) = f + g = \Delta(f)(i) + \Delta(g)(i)$, and Δ is additive too. Thus we may conclude that colim is right exact.

To see that pushout, the colimit of $\begin{smallmatrix} \bullet & \to & \bullet \\ \downarrow & & \\ \bullet & & \end{smallmatrix}$, is not exact in **Ab**, we will give an explicit example. Let I be the category $\begin{smallmatrix} \bullet & \to & \bullet \\ \downarrow & & \\ \bullet & & \end{smallmatrix}$. Let $0 \to F \to G \to H \to 0$ be exact in $\mathcal{A}^I$, i.e., for all $\bullet \in I$, $0 \to F(\bullet) \to G(\bullet) \to H(\bullet) \to 0$ is exact in $\mathcal{A}$. Consider the example

$$0 \to \begin{matrix} \mathbf{Z} & \longrightarrow & 0 \\ \big\downarrow {\scriptstyle \cdot p} & & \\ p\mathbf{Z} & & \end{matrix} \xrightarrow{\begin{smallmatrix} \iota_1 & 0 \\ \cdot p & \end{smallmatrix}} \begin{matrix} \mathbf{Z} \oplus \mathbf{Z} & \xrightarrow{\pi} & \mathbf{Z} \\ \big\downarrow {\scriptstyle \pi} & & \\ \mathbf{Z} & & \end{matrix} \xrightarrow{\begin{smallmatrix} \pi_2 & \mathrm{id} \\ [-] & \end{smallmatrix}} \begin{matrix} \mathbf{Z} & \xrightarrow{\mathrm{id}} & \mathbf{Z} \\ \big\downarrow & & \\ \mathbf{Z}/p\mathbf{Z} & & \end{matrix} \to 0.$$

Then we have the pushouts

$$\begin{array}{ccc}
\begin{tikzcd} \mathbf{Z} \arrow[r] \arrow[d] & 0 \arrow[d] \\ p\mathbf{Z} \arrow[r] & 0 \end{tikzcd}
&
\begin{tikzcd} \mathbf{Z}\oplus\mathbf{Z} \arrow[r] \arrow[d] & \mathbf{Z} \arrow[d] \\ \mathbf{Z} \arrow[r] & \mathbf{Z} \end{tikzcd}
&
\begin{tikzcd} \mathbf{Z} \arrow[r] \arrow[d] & \mathbf{Z} \arrow[d] \\ \mathbf{Z}/p\mathbf{Z} \arrow[r] & \mathbf{Z}/p\mathbf{Z} \end{tikzcd}
\end{array}$$

And the sequence $0 \to 0 \to \mathbf{Z} \to \mathbf{Z}/p\mathbf{Z} \to 0$ is not exact, since $\mathbf{Z} \ncong \mathbf{Z}/p\mathbf{Z}$.

Proposition 2.6.8 *The following are equivalent for an abelian category $\mathcal{A}$:*

1. *The direct sum $\oplus A_i$ exists in $\mathcal{A}$ for every set $\{A_i\}$ of objects in $\mathcal{A}$.*
2. *$\mathcal{A}$ is* cocomplete*, that is,* $\operatorname{colim}_{i\in I} A_i$ *exists in $\mathcal{A}$ for each functor $A : I \to \mathcal{A}$ whose indexing category I has only a set of objects.*

Proof. As (1) is a special case of (2), we assume (1) and prove (2). Given $A : I \to \mathcal{A}$, the cokernel C of

$$\bigoplus_{\varphi: i\to j} A_i \to \bigoplus_{i\in I} A_i$$
$$a_i[\varphi] \mapsto \varphi(a_i) - a_i$$

solves the universal problem defining the colimit, so $C = \operatorname*{colim}_{i\in I} A_i$. ☐

Remark **Ab**, **mod** $- R$, Presheaves(X), and Sheaves(X) are cocomplete because (1) holds. (If I is infinite, the direct sum in Sheaves(X) is the sheafification of the direct sum in Presheaves(X).) The category of finite abelian groups has only *finite* direct sums, so it is not cocomplete.

Variation 2.6.9 (Limits) The limit of a functor $A : I \to \mathcal{A}$ is the colimit of the corresponding functor $A^{op} : I^{op} \to \mathcal{A}^{op}$, so all the above remarks apply in dual form to limits. In particular, $\lim : \mathcal{A}^I \to \mathcal{A}$ is right adjoint to the diagonal functor Δ, so lim is a left exact functor when it exists. If the product $\prod A_i$ of every set $\{A_i\}$ of objects exists in $\mathcal{A}$, then $\mathcal{A}$ is *complete*, that is, $\lim_{i\in I} A_i$ exists for every $A : I \to \mathcal{A}$ with I having only a set of objects. **Ab**, **mod** $- R$, Presheaves(X), and Sheaves(X) are complete because such products exists.

One of the most useful properties of adjoint functors is the following result, which we quote without proof from [MacCW,V.5].

Adjoints and Limits Theorem 2.6.10 *Let $L : \mathcal{A} \to \mathcal{B}$ be left adjoint to a functor $R : \mathcal{B} \to \mathcal{A}$, where $\mathcal{A}$ and $\mathcal{B}$ are arbitrary categories. Then*

1. *L preserves all colimits (coproducts, direct limits, cokernels, etc.). That is, if $A : I \to \mathcal{A}$ has a colimit, then so does $LA : I \to \mathcal{B}$, and*
$$L(\operatorname*{colim}_{i\in I} A_i) = \operatorname*{colim}_{i\in I} L(A_i).$$
2. *R preserves all limits (products, inverse limits, kernels, etc.). That is, if $B : I \to \mathcal{B}$ has a limit, then so does $RB : I \to \mathcal{A}$, and*
$$R(\lim_{i\in I} B_i) = \lim_{i\in I} R(B_i).$$

We say that $\mathcal{A}$ satisfies axiom (AB4) if it is cocomplete and direct sums of monics are monic, i.e., homology commutes with direct sums. This is true for **Ab** and **mod** $- R$. (Homology does not commute with arbitrary colimits; the derived functors of colim intervene via a spectral sequence.) Here are two consequences of axiom (AB4).

Corollary 2.6.11 *If a abelian category $\mathcal{A}$ satisfying (AB4) has enough projectives, and $F : \mathcal{A} \to \mathcal{B}$ is a left adjoint, then for every set $\{A_i\}$ of objects in $\mathcal{A}$:*

$$L_*F\left(\bigoplus_{i\in I} A_i\right) \cong \bigoplus_{i\in I} L_*F(A_i).$$

Proof. If $P_i \to A_i$ are projective resolutions, then so is $\oplus P_i \to \oplus A_i$. Hence

$$L_*F(\oplus A_i) = H_*(F(\oplus P_i)) \cong H_*(\oplus F(P_i)) \cong \oplus H_*(F(P_i)) = \oplus L_*F(A_i).$$

□

Corollary 2.6.12 $\operatorname{Tor}_*(A, \oplus_{i\in I} B_i) \cong \oplus_{i\in I} \operatorname{Tor}_*(A, B_i)$.

Proof. If $P \to A$ is a projective resolution, then

$$\begin{aligned}\operatorname{Tor}_*(A, \oplus B_i) &= H_*(P \otimes (\oplus B_i)) \cong H_*(\oplus(P \otimes B_i)) \cong \oplus H_*(P \otimes B_i)\\ &= \oplus \operatorname{Tor}_*(A, B_i).\end{aligned}$$

□

Definition 2.6.13 A nonempty category I is called *filtered* if

1. For every $i, j \in I$ there are arrows $\begin{smallmatrix} i \searrow \\ \quad k \\ j \nearrow \end{smallmatrix}$ to some $k \in I$.
2. For every two parallel arrows $u, v : i \rightrightarrows j$ there is an arrow $w : j \to k$ such that $wu = wv$.

A *filtered colimit* in $\mathcal{A}$ is just the colimit of a functor $A : I \to \mathcal{A}$ in which I is a filtered category. We shall use the notation $\underset{\to}{\operatorname{colim}}(A_i)$ for such a filtered colimit.

If I is a partially ordered set (poset), considered as a category, then condition (2) always holds, and (1) just requires that every pair of elements has an upper bound in I. A filtered poset is often called *directed*; filtered colimits over directed posets are often called *direct limits* and are often written $\underset{\to}{\lim} A_i$.

We are going to show that direct limits and filtered colimits of modules are exact. First we obtain a more concrete description of the elements of $\underset{\to}{\operatorname{colim}}(A_i)$.

Lemma 2.6.14 *Let I be a filtered category and $A : I \to \mathbf{mod} - R$ a functor. Then*

1. *Every element $a \in \underset{\to}{\operatorname{colim}}(A_i)$ is the image of some element $a_i \in A_i$ (for some $i \in I$) under the canonical map $A_i \to \underset{\to}{\operatorname{colim}}(A_i)$.*
2. *For every i, the kernel of the canonical map $A_i \to \underset{\to}{\operatorname{colim}}(A_i)$ is the union of the kernels of the maps $\varphi : A_i \to A_j$ (where $\varphi : i \to j$ is a map in I).*

Proof. We shall use the explicit construction of $\underset{\to}{\operatorname{colim}}(A_i)$. Let $\lambda_i : A_i \to \oplus_{i\in I} A_i$ be the canonical maps. Every element a of $\underset{\to}{\operatorname{colim}} A_i$ is the image of

$$\sum_{j\in J} \lambda_j(a_j)$$

for some finite set $J = \{i_1, \cdots, i_n\}$. There is an upper bound i in I for $\{i_1, \cdots, i_n\}$; using the maps $A_j \to A_i$ we can represent each a_j as an element in A_i and take a_i to be their sum. Evidently, a is the image of a_i, so (1) holds.

Now suppose that $a_i \in A_i$ vanishes in $\underrightarrow{\operatorname{colim}}(A_i)$. Then there are $\varphi_{jk} : j \to k$ in I and $a_j \in A_j$ so that $\lambda_i(a_i) = \sum \lambda_k(\varphi_{jk}(a_j)) - \lambda_j(a_j)$ in $\oplus A_i$. Choose an upper bound t in I for all the i, j, k in this expression. Adding $\lambda_t(\varphi_{it} a_i) - \lambda_i(a_i)$ to both sides we may assume that $i = t$. Adding zero terms of the form

$$[\lambda_t \varphi_{jt}(a_j) - \lambda_k \varphi_{jk}(a_j)] + [\lambda_t \varphi_{jt}(-a_j) - \lambda_k \varphi_{jk}(-a_j)],$$

we can assume that the k's are t. If any φ_{jt} are parallel arrows in I, then by changing t we can equalize them. Therefore we have

$$\lambda_t(a_t) = \lambda_t\left(\sum \varphi_{jt}(a_j)\right) - \sum \lambda_j(a_j)$$

with all the j's distinct and none equal to t. Since the λ_j are injections, all the a_j must be zero. Hence $\varphi_{it}(a_i) = a_t = 0$, that is, $a_i \in \ker(\varphi_{it})$. $\square$

Theorem 2.6.15 *Filtered colimits (and direct limits) of R-modules are exact, considered as functors from $(\mathbf{mod} - R)^I$ to $\mathbf{mod} - R$.*

Proof. Set $\mathcal{A} = \mathbf{mod} - R$. We have to show that if I is a filtered category (*e.g.*, a directed poset), then $\underrightarrow{\operatorname{colim}} : \mathcal{A}^I \to \mathcal{A}$ is exact. Exercise 2.6.4 showed that $\underrightarrow{\operatorname{colim}}$ is right exact, so we need only prove that if $t : A \to B$ is monic in $\mathcal{A}^I$ (*i.e.*, each t_i is monic), then $\underrightarrow{\operatorname{colim}}(A_i) \to \underrightarrow{\operatorname{colim}}(B_i)$ is monic in $\mathcal{A}$. Let $a \in \underrightarrow{\operatorname{colim}}(A_i)$ be an element that vanishes in $\underrightarrow{\operatorname{colim}}(B_i)$. By the lemma above, a is the image of some $a_i \in A_i$. Therefore $t_i(a_i) \in B_i$ vanishes in $\underrightarrow{\operatorname{colim}}(B_i)$, so there is some $\varphi : i \to j$ so that

$$0 = \varphi(t_i(a_i)) = t_j(\varphi(a_i)) \text{ in } B_j.$$

Since t_j is monic, $\varphi(a_i) = 0$ in A_j. Hence $a = 0$ in $\underrightarrow{\operatorname{colim}}(A_i)$. $\square$

Exercise 2.6.5 (AB5) The above theorem does not hold for every cocomplete abelian category $\mathcal{A}$. Show that if $\mathcal{A}$ is the opposite category $\mathbf{Ab}^{op}$ of abelian groups, then the functor $\underrightarrow{\operatorname{colim}} : \mathcal{A}^I \to \mathcal{A}$ need not be exact when I is filtered.
An abelian category $\mathcal{A}$ is said to satisfy axiom (AB5) if it is cocomplete and filtered colimits are exact. Thus the above theorem states that $\mathbf{mod} - R$ and $R - \mathbf{mod}$ satisfy axiom (AB5), and this exercise shows that $\mathbf{Ab}^{op}$ does not.

We produce an explicit example. Let I be the poset category $\bullet \to \bullet \to \bullet \to \cdots$, and let $\mathcal{A} = \mathbf{Ab}^{op}$. Then the filtered colimit $\underrightarrow{\operatorname{colim}} : \mathcal{A}^I \to \mathcal{A}$ is the inverse limit of $G_0 \leftarrow G_1 \leftarrow G_2 \leftarrow \cdots$. Let $0 \to F \to G \to H \to 0$ be exact in $\mathcal{A}^I$, i.e., exact for all $\bullet \in I$. Our example is as follows:

$$\begin{tikzcd}
0 \arrow[r] & p\mathbf{Z} \arrow[r, hook] & \mathbf{Z} \arrow[r, two heads] & \mathbf{Z}/p\mathbf{Z} \arrow[r] & 0 \\
0 \arrow[r] & p^2\mathbf{Z} \arrow[r, hook] \arrow[u] & \mathbf{Z} \arrow[r, two heads] \arrow[u, "\mathrm{id}"'] & \mathbf{Z}/p^2\mathbf{Z} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & p^3\mathbf{Z} \arrow[r, hook] \arrow[u] & \mathbf{Z} \arrow[r, two heads] \arrow[u, "\mathrm{id}"'] & \mathbf{Z}/p^3\mathbf{Z} \arrow[r] \arrow[u] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

Then, observe that the inverse limit of column three is the p-adics, $\mathbf{Z}_p$. The inverse limit of column two is $\mathbf{Z}$. Yet $\mathbf{Z} \to \mathbf{Z}_p \to 0$ is not exact, because $\mathbf{Z}_p \not\subseteq \mathbf{Z}$. Thus filtered colimits in $\mathbf{Ab}^{op}$ are not necessarily exact, and hence $\mathbf{Ab}^{op}$ does not satisfy (AB5).

Exercise 2.6.6 Let $f : X \to Y$ be a continuous map. Show that the inverse image sheaf functor f^{-1} : Sheaves$(Y) \to$ Sheaves(X) is exact. (See 2.6.6.)

Let $\mathcal{G}$ be a sheaf on Y. Recall that $f^{-1}\mathcal{G}$ is a sheaf on X defined by the sheafification of the presheaf defined for all $U \subseteq X$ by $\mathcal{P}(U) = \varinjlim_{f(U)\subseteq V\subseteq Y} \mathcal{G}(V)$. By remark in Application 2.6.5, sheafification is an exact functor, so we may check exactness on presheaves without concern.

To show that the inverse image functor is exact, we show exactness on the level of stalks of sheaves; i.e., $0 \to \mathcal{G} \to \mathcal{G}' \to \mathcal{G}'' \to 0$ is exact in Sheaves(Y) if and only if the sequence $0 \to \mathcal{G}_y \to \mathcal{G}'_y \to \mathcal{G}''_y \to 0$ is exact in $\mathbf{Ab}$ for $y \in Y$.

We first claim that if $x \in X$, then there is an isomorphism $\mathcal{G}_{f(x)} \cong (f^{-1}\mathcal{G})_x$. To see this, observe that by definition of stalk, $\mathcal{G}_{f(x)} = \varinjlim_{f(x)\in V} \mathcal{G}(V)$, while

$$(f^{-1}\mathcal{G})_x = \varinjlim_{x\in U} f^{-1}\mathcal{G}(U) = \varinjlim_{x\in U} \varinjlim_{f(U)\subseteq V} \mathcal{G}(V).$$

Now notice that for fixed $x \in X$, taking a limit of smaller and smaller neighborhoods V around $f(x)$ is in one-to-one correspondence with taking a limit of smaller and smaller neighborhoods U around x and taking their image $f(U)$ around $f(x)$. Thus

$$\mathcal{G}_{f(x)} = \varinjlim_{f(x)\in V} \mathcal{G}(V) = \varinjlim_{x\in U} \varinjlim_{f(U)\subseteq V} \mathcal{G}(V),$$

as we claimed. With the claim shown, exactness is now evident. Let $0 \to \mathcal{G} \to \mathcal{G}' \to \mathcal{G}'' \to 0$ be exact in Sheaves(Y). This implies the sequence $0 \to \mathcal{G}_{f(x)} \to \mathcal{G}'_{f(x)} \to \mathcal{G}''_{f(x)} \to 0$ is exact in $\mathbf{Ab}$, and by the isomorphism, $0 \to \left(f^{-1}\mathcal{G}\right)_x \to \left(f^{-1}\mathcal{G}'\right)_x \to \left(f^{-1}\mathcal{G}''\right)_x \to 0$ is exact in $\mathbf{Ab}$, so $0 \to f^{-1}\mathcal{G} \to f^{-1}\mathcal{G}' \to f^{-1}\mathcal{G}'' \to 0$ is exact in Sheaves(X), as we wished to show.

The following consequences are proven in the same manner as their counterparts for direct sum. Note that in categories like $R - \mathbf{mod}$ for which filtered colimits are exact, homology commutes with filtered colimits.

Corollary 2.6.16 *Suppose that $\mathcal{A} = R - \mathbf{mod}$ and $\mathcal{B} = \mathbf{Ab}$ (or $\mathcal{A}$ is any abelian category with enough projectives, and $\mathcal{A}$ and $\mathcal{B}$ satisfy axiom (AB5)). If $F : \mathcal{A} \to \mathcal{B}$ is a left adjoint, then for every $A : I \to \mathcal{A}$*

with I filtered

$$L_*F(\underset{\rightarrow}{\operatorname{colim}}(A_i)) \cong \underset{\rightarrow}{\operatorname{colim}}\, L_*F(A_i).$$

Corollary 2.6.17 *For every filtered $B : I \to R-\mathbf{mod}$ and every $A \in \mathbf{mod}-R$,*

$$\operatorname{Tor}_*(A, \underset{\rightarrow}{\operatorname{colim}}(B_i)) \cong \underset{\rightarrow}{\operatorname{colim}}\, \operatorname{Tor}_*(A, B_i).$$

2.7 Balancing Tor and Ext

In earlier sections we promised to show that the two left derived functors of $A \otimes_R B$ gave the same result and that the two right derived functors of $\operatorname{Hom}(A, B)$ gave the same result. It is time to deliver on these promises.

Tensor Product of Complexes 2.7.1 Suppose that P and Q are chain complexes of right and left R-modules, respectively. Form the double complex $P \otimes_R Q = \{P_p \otimes_R Q_q\}$ using the sign trick, that is, with horizontal differentials $d \otimes 1$ and vertical differentials $(-1)^p \otimes d$. $P \otimes_R Q$ is called the *tensor product double complex*, and $\operatorname{Tot}^{\oplus}(P \otimes_R Q)$ is called the *(total) tensor product chain complex* of P and Q.

Theorem 2.7.2 *$L_n(A \otimes_R)(B) \cong L_n(\otimes_R B)(A) = \operatorname{Tor}_n^R(A, B)$ for all n.*

Proof. Choose a projective resolution $P \xrightarrow{\varepsilon} A$ in **mod**-R and a projective resolution $Q \xrightarrow{\eta} B$ in R-**mod**. Thinking of A and B as complexes concentrated in degree zero, we can form the three tensor product double complexes $P \otimes Q$, $A \otimes Q$, and $P \otimes B$. The augmentations ε and η induce maps from $P \otimes Q$ to $A \otimes Q$ and $P \otimes B$.

$$\begin{tikzcd}
\vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & & \\
A \otimes Q_2 \arrow[d] & & P_0 \otimes Q_2 \arrow[d, "d"'] & P_1 \otimes Q_2 \arrow[l, "d"'] \arrow[d, "-d"'] & \cdots \arrow[l, "d"'] \arrow[d] & \\
A \otimes Q_1 \arrow[d] & & P_0 \otimes Q_1 \arrow[d, "d"'] & P_1 \otimes Q_1 \arrow[l, "d"'] \arrow[d, "-d"'] & P_2 \otimes Q_1 \arrow[l, "d"'] \arrow[d, "d"'] & \cdots \arrow[l] \\
A \otimes Q_0 & & P_0 \otimes Q_0 & P_1 \otimes Q_0 \arrow[l, "d"'] & P_2 \otimes Q_0 \arrow[l, "d"'] & \cdots \arrow[l] \\
 & & & & & \\
 & & P_0 \otimes B & P_1 \otimes B \arrow[l, "d"'] & P_2 \otimes B \arrow[l, "d"'] & \cdots \arrow[l]
\end{tikzcd}$$

Using the Acyclic Assembly Lemma 2.7.3, we will show that the maps

$$A \otimes Q = \operatorname{Tot}(A \otimes Q) \xleftarrow{\varepsilon \otimes Q} \operatorname{Tot}(P \otimes Q) \xrightarrow{P \otimes \eta} \operatorname{Tot}(P \otimes B) = P \otimes B$$

are quasi-isomorphisms, inducing the promised isomorphisms on homology:

$$L_*(A \otimes_R)(B) \xleftarrow{\cong} H_*(\operatorname{Tot}(P \otimes Q)) \xrightarrow{\cong} L_*(\otimes_R B)(A).$$

Consider the double complex C obtained from $P \otimes Q$ by adding $A \otimes Q[-1]$ in the column $p = -1$. The translate $\operatorname{Tot}(C)[1]$ is the mapping cone of the map $\varepsilon \otimes Q$ from $\operatorname{Tot}(P \otimes Q)$ to $A \otimes Q$ (see 1.2.8 and 1.5.1), so in order to show that $\varepsilon \otimes Q$ is a quasi-isomorphism, it suffices to show that $\mathrm{Tot}(C)$ is acyclic. Since each $\otimes Q_q$ is an exact functor, every row of C is exact, so $\mathrm{Tot}(C)$ is exact by the Acyclic Assembly Lemma.

Similarly, the mapping cone of $P \otimes \eta : \mathrm{Tot}(P \otimes Q) \to P \otimes B$ is the translate $\mathrm{Tot}(D)[1]$, where D is the double complex obtained from $P \otimes Q$ by adding $P \otimes B[-1]$ in the row $q = -1$. Since each $P_p \otimes$ is an exact functor, every column of D is exact, so $\mathrm{Tot}(D)$ is exact by the Acyclic Assembly Lemma 2.7.3. Hence $\mathrm{cone}(P \otimes \eta)$ is acyclic, and $P \otimes \eta$ is also a quasi-isomorphism. $\square$

Acyclic Assembly Lemma 2.7.3 *Let C be a double complex in* **mod**-*R. Then*

- $\mathrm{Tot}^{\prod}(C)$ *is an acyclic chain complex, assuming either of the following:*
  1. *C is an upper half-plane complex with exact columns.*
  2. *C is a right half-plane complex with exact rows.*
- $\mathrm{Tot}^{\oplus}(C)$ *is an acyclic chain complex, assuming either of the following:*
  3. *C is an upper half-plane complex with exact rows.*
  4. *C is a right half-plane complex with exact columns.*

Remark The proof will show that in (1) and (3) it suffices to have every diagonal bounded on the lower right, and in (2) and (4) it suffices to have every diagonal bounded on the upper left. See 5.5.1 and 5.5.10.

Proof. We first show that it suffices to establish case (1). Interchanging rows and columns also interchanges (1) and (2), and (3) and (4), so (1) implies (2) and (4) implies (3). Suppose we are in case (4), and let $\tau_n C$ be the double subcomplex of C obtained by truncating each column at level n:

$$(\tau_n C)_{p,q} \begin{cases} C_{p,q} & \text{if } q > n \\ \ker(d^v : C_{p,n} \to C_{p,n-1}) & \text{if } q = n \\ 0 & \text{if } q < n \end{cases} .$$

Each $\tau_n C$ is, up to vertical translation, a first quadrant double complex with exact columns, so (1) implies that $\mathrm{Tot}^{\oplus}(\tau_n C) = \mathrm{Tot}^{\prod}(\tau_n C)$ is acyclic. This implies that $\mathrm{Tot}^{\oplus}(C)$ is acyclic, because every cycle of $\mathrm{Tot}^{\oplus}(C)$ is a cycle (hence a boundary) in some subcomplex $\mathrm{Tot}^{\oplus}(\tau_n C)$. Therefore (1) implies (4) as well.

In case (1), translating C left and right, suffices to prove that $H_0(\mathrm{Tot}(C))$ is zero. Let

$$c = (\cdots, c_{-p,p}, \cdots, c_{-2,2}, c_{-1,1}, c_{0,0}) \in \prod C_{-p,p} = \mathrm{Tot}(C)_0$$

be a 0-cycle; we will find elements $b_{-p,p+1}$ by induction on p so that

$$d^v(b_{-p,p+1}) + d^h(b_{-p+1,p}) = c_{-p,p}.$$

Assembling the b's will yield an element b of $\prod C_{-p,p+1}$ such that $d(b) = c$, proving that $H_0(\mathrm{Tot}(C)) = 0$. The following schematic should help give the idea.

$$
\begin{tikzcd}
b_{-p,p+1} \arrow[d, mapsto] & & & & & \\
c_{-p,p} & b_{-p+1,p} \arrow[l, mapsto] \arrow[d, mapsto] & & & & \\
& c_{-p+1,p-1} & b_{-p+2,p-1} \arrow[l, mapsto] & & & \\
& & \vdots \arrow[d, mapsto] & & & \\
& & c_{-2,2} & b_{-1,2} \arrow[l, mapsto] \arrow[d, mapsto] & & \\
& & & c_{-1,1} & b_{0,1} \arrow[l, mapsto] \arrow[d, mapsto] & \\
& & & & c_{0,0} \arrow[d, mapsto] & 0(=b_{1,0}) \arrow[l, mapsto] \\
& & & & 0 &
\end{tikzcd}
$$

We begin the induction by choosing $b_{1,0} = 0$ for $p = -1$. Since $c_{0,-1} = 0$, $d^v(c_{0,0}) = 0$; since the 0^{th} column is exact, there is a $b_{0,1} \in C_{0,1}$ so that $d^v(b_{0,1}) = c_{0,0}$. Inductively, we compute that

$$
\begin{aligned}
d^v(c_{-p,p} - d^h(b_{-p+1,p})) &= d^v(c_{-p,p}) + d^h d^v(b_{-p+1,p}) \\
&= d^v(c_{-p,p}) + d^h(c_{-p+1,p-1}) - d^h d^h(b_{-p+2,p-1}) \\
&= 0.
\end{aligned}
$$

Since the $-p^{th}$ column is exact, there is a $b_{-p,p+1}$ so that

$$d^v(b_{-p,p+1}) = c_{-p,p} - d^h(b_{-p+1,p})$$

as desired. $\square$

Exercise 2.7.1 Let C be the periodic upper half-plane complex with $C_{p,q} = \mathbf{Z}/4$ for all p and $q \geq 0$, all differentials being multiplication by 2.

$$
\begin{tikzcd}
& {} \arrow[d, "2"] & {} \arrow[d, "2"] & {} \arrow[d, "2"] & \\
\cdots & \mathbf{Z}/4 \arrow[l, "2"'] \arrow[d, "2"] & \mathbf{Z}/4 \arrow[l, "2"'] \arrow[d, "2"] & \mathbf{Z}/4 \arrow[l, "2"'] \arrow[d, "2"] & \cdots \arrow[l, "2"'] \\
\cdots & \mathbf{Z}/4 \arrow[l, "2"'] & \mathbf{Z}/4 \arrow[l, "2"'] & \mathbf{Z}/4 \arrow[l, "2"'] & \cdots \arrow[l, "2"']
\end{tikzcd}
$$

1. Show that $H_0(\mathrm{Tot}^{\prod}(C)) \cong \mathbf{Z}/2$ on the cycle $(\ldots, 1, 1, 1) \in \prod C_{-p,p}$ even though the rows of C are exact. *Hint*: First show that the 0-boundaries are $\prod 2\mathbf{Z}/4$.
2. Show that $\mathrm{Tot}^{\oplus}(C)$ is acyclic.
3. Now extend C downward to form a doubly periodic plane double complex D with $D_{p,q} = \mathbf{Z}/4$ for all $p, q \in \mathbf{Z}$. Show that $H_0(\mathrm{Tot}^{\prod}(D))$ maps onto $H_0(\mathrm{Tot}^{\prod} C) \cong \mathbf{Z}/2$. Hence $\mathrm{Tot}^{\prod}(D)$ is not acyclic, even though every row and column of D is exact. Show that $\mathrm{Tot}^{\oplus}(D)$ is not acyclic either.

Recall that $\mathrm{Tot}^{\Pi}(C)_n = \mathrm{Tot}(C)_n = \prod\limits_{p+q=n} C_{p,q}$ and $\mathrm{Tot}^{\oplus}(C)_n = \bigoplus\limits_{p+q=n} C_{p,q}$, both with differential $d = d^h + d^v$.

1. To see that $H_0(\mathrm{Tot}(C)) \cong \mathbf{Z}/2\mathbf{Z}$, we follow the hint. We first show that $B_0 = \operatorname{im} d_1 \cong \prod 2\mathbf{Z}/4\mathbf{Z}$. Let $(..., c_{-1,2}, c_{0,1}, c_{1,0}) \in \prod\limits_{p+q=1} C_{p,q}$; then

$$\begin{aligned} d(..., c_{-1,2}, c_{0,1}, c_{1,0}) &= d^h(..., c_{-1,2}, c_{0,1}, c_{1,0}) + d^v(..., c_{-1,2}, c_{0,1}, c_{1,0}) \\ &= (..., 2c_{-1,2}, 2c_{0,1}, 2c_{1,0}) + (..., 2c_{-2,3}, 2c_{-1,2}, 2c_{0,1}) \\ &= (..., 2(c_{-1,2} + c_{-2,3}), 2(c_{0,1} + c_{-1,2}), 2(c_{1,0} + c_{0,1})). \end{aligned}$$

   Thus, $B_0 \cong \prod 2\mathbf{Z}/4\mathbf{Z}$, as claimed.

   Next, we compute $Z_0 = \ker d_0$. See that for $(..., c_{-2,2}, c_{-1,1}, c_{0,0}) \in \prod\limits_{p+q=0} C_{p,q}$,

$$d(..., c_{-2,2}, c_{-1,1}, c_{0,0}) = 2(..., c_{-2,2} + c_{-3,3}, c_{-1,1} + c_{-2,2}, c_{0,0} + c_{-1,1})$$

   is equal to zero if and only if

$$\begin{aligned} c_{0,0} + c_{-1,1} &\text{ is even,} \\ c_{-1,1} + c_{-2,2} &\text{ is even,} \\ c_{-2,2} + c_{-3,3} &\text{ is even,} \\ &\vdots \end{aligned}$$

   Thus, all $\{c_{-p,p}\}$ must have the same parity. This means an element of $\ker d_0$ is first a choice of even or odd element for $c_{0,0}$, and then for all subsequent $c_{-p,p}$, a choice of two elements: either 0 or 2, or 1 or 3. Therefore, $Z_0 \cong \prod 2\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$.

   We can conclude that $H_0(\mathrm{Tot}(C)) = \left(\prod 2\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}\right) / \left(\prod 2\mathbf{Z}/4\mathbf{Z}\right) \cong \mathbf{Z}/2\mathbf{Z}$, as we wished to show.

2. C is an upper half-plane complex with exact rows, so $\mathrm{Tot}^{\oplus}(C)$ is acyclic by the Acyclic Assembly Lemma 2.7.3. Indeed, the rows are exact, as

$$\operatorname{im}\left(\mathbf{Z}/4\mathbf{Z} \xrightarrow{2} \mathbf{Z}/4\mathbf{Z}\right) = \mathbf{Z}/2\mathbf{Z} = \ker\left(\mathbf{Z}/4\mathbf{Z} \xrightarrow{2} \mathbf{Z}/4\mathbf{Z}\right).$$

3. Since $(..., 1, 1, 1)$ is a nonzero cycle in $\mathrm{Tot}^{\prod}(C)_0$, $(..., 1, 1, 1, 0, 0, ...)$ is a nonzero cycle in $\mathrm{Tot}^{\prod}(D)_0$, so $H_0(\mathrm{Tot}(D)) \twoheadrightarrow H_0(\mathrm{Tot}(C)) \cong \mathbf{Z}/2\mathbf{Z}$, and thus $H_0(\mathrm{Tot}(D)) \ncong 0$. To see that $H_0(\mathrm{Tot}^{\oplus}(D))$ is not zero either, it is enough to show there is a nonzero element, i.e., a cycle that is not a boundary. We claim such an element is $(..., 0, 2, 0, ...) \in \mathrm{Tot}^{\oplus}(D)_0$. To see this, observe that $(..., 0, 2, 0...) \in \ker d$, since

$$d(..., 0, 2, 0, ...) = (..., 0, 4, 4, 0, ...) = 0,$$

but $(..., 0, 2, 0, ...) \notin \operatorname{im} d$. Indeed, for an element $(..., x_1, x_0, x_{-1}, ...) \in \mathrm{Tot}^{\oplus}(D)_1$,

$$d(..., x_1, x_0, x_{-1}, ...) = 2(..., x_1 + x_2, x_0 + x_1, x_{-1} + x_0, ...)$$

is equal to $(..., 0, 2, 0, ...)$ when $x_0 + x_1 = 1$ and $x_i + x_{i+1}$ is even for all $i \neq 0$. As $x_0 + x_1 = 1$, x_0 and x_1 must be of opposite parity. Without loss of generality, x_1 is odd. Observe that

$$\begin{gathered} x_1 + x_2 \text{ is even, so } x_2 \text{ must also be odd,} \\ x_2 + x_3 \text{ is even, so } x_3 \text{ must also be odd,} \\ x_3 + x_4 \text{ is even, so } x_4 \text{ must also be odd,} \\ \vdots \end{gathered}$$

Thus $\{x_n\}$ are odd for $n \in \mathbf{N}$. Odd elements must not be zero, so $(..., x_1, x_0, x_{-1}, ...)$ has infinitely many nonzero entries, a contradiction, as a direct sum must have all but finitely many zero. Thus, $(..., 0, 2, 0, ...) \notin \operatorname{im} d$, and therefore there is a nonzero element of $H_0(\mathrm{Tot}^{\oplus}(D))$, as desired.

Exercise 2.7.2

1. Give an example of a 2^{nd} quadrant double chain complex C with exact columns for which $\mathrm{Tot}^{\oplus}(C)$ is not an acyclic chain complex.
2. Give an example of a 4^{th} quadrant double complex C with exact columns for which $\mathrm{Tot}^{\prod}(C)$ is not acyclic.

1. Consider the double complex C given by the following diagram:

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\cdots & \mathbf{Z} \arrow[l] \arrow[d] & \mathbf{Z} \arrow[l, "2"'] \arrow[d, "\mathrm{id}"'] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] \\
\cdots & 0 \arrow[l] \arrow[d] & \mathbf{Z} \arrow[l] \arrow[d] & \mathbf{Z} \arrow[l, "2"'] \arrow[d, "\mathrm{id}"'] & 0 \arrow[l] \arrow[d] \\
\cdots & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \mathbf{Z} \arrow[l] \arrow[d] & \mathbf{Z} \arrow[l, "2"'] \arrow[d, "\mathrm{id}"'] \\
 & 0 \arrow[l] & 0 \arrow[l] & 0 \arrow[l] & \mathbf{Z} \arrow[l] \\
 & & & & C_{0,0}
\end{tikzcd}
$$

C is a double complex, as evaluating on every square yields 0, so $d^h d^v = -d^v d^h = 0$. C is nonzero only in the second quadrant. Every column is exact. To see that $\mathrm{Tot}^{\oplus}(C)$ is not acyclic, we give a cycle that is not a boundary. First, observe that the only nontrivial differential in $\mathrm{Tot}^{\oplus}(C)$ is the map $d : \bigoplus_{p+q=1} C_{p,q} \to \bigoplus_{p+q=0} C_{p,q}$ defined by

$$
\begin{aligned}
d(..., x_3, x_2, x_1, x_0) &= d^v(..., x_3, x_2, x_1, x_0) + d^h(..., x_3, x_2, x_1, x_0) \\
&= (..., x_3, x_2, x_1, x_0) + (..., 2x_2, 2x_1, 2x_0, 0) \\
&= (..., x_3 + 2x_2, x_2 + 2x_1, x_1 + 2x_0, x_0).
\end{aligned}
$$

Now, see that the element $(..., 0, 0, 1) \in \bigoplus_{p+q=0} C_{p,q}$ is obviously a cycle, as all elements in $\bigoplus_{p+q=0} C_{p,q}$ map to 0, but $(..., 0, 0, 1)$ is not in the image of d and hence not a boundary. Indeed, see that if $d(..., x_1, x_0) = (..., x_1 + 2x_0, x_0) = (..., 0, 1)$, then

$$
\begin{aligned}
x_0 &= 1, \\
x_1 + 2x_0 = 0;\ \text{i.e., } x_1 &= -2, \\
x_2 + 2x_1 = 0;\ \text{i.e., } x_2 &= 4, \\
x_3 + 2x_2 = 0;\ \text{i.e., } x_3 &= -8, \\
&\vdots
\end{aligned}
$$

Inductively, $x_i = (-2)^i$. But such an expression $(..., -8, 4, -2, 1)$ must have infinitely many nonzero terms and hence cannot be an element of $\bigoplus_{p+q=1} C_{p,q}$. Thus, $(..., 0, 1)$ is not a boundary, and $\mathrm{Tot}^{\oplus}(C)$ is not acyclic, as desired.

2. A very similar construction for C follows:

$$C_{0,0}$$

$$\begin{tikzcd}
\mathbf{Z} \arrow[d, "\mathrm{id}"'] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \arrow[l] \\
\mathbf{Z} \arrow[d] & \mathbf{Z} \arrow[l, "2"'] \arrow[d, "\mathrm{id}"'] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
0 \arrow[d] & \mathbf{Z} \arrow[l] \arrow[d] & \mathbf{Z} \arrow[l, "2"'] \arrow[d, "\mathrm{id}"'] & 0 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
0 \arrow[d] & 0 \arrow[l] \arrow[d] & \mathbf{Z} \arrow[l] \arrow[d] & \mathbf{Z} \arrow[l, "2"'] \arrow[d] & \cdots \arrow[l] \\
\vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

The nontrivial differential is $d : \prod_{p+q=0} C_{p,q} \to \prod_{p+q=-1} C_{p,q}$ defined by

$$\begin{aligned}
d(x_0, x_1, x_2, x_3, ...) &= d^v(x_0, x_1, x_2, x_3, ...) + d^h(x_0, x_1, x_2, x_3, ...) \\
&= (x_0, x_1, x_2, x_3, ...) + (2x_1, 2x_2, 2x_3, 2x_4, ...) \\
&= (x_0 + 2x_1, x_1 + 2x_2, x_2 + 2x_3, x_3 + 2x_4, ...).
\end{aligned}$$

The element $(1, 1, 1, ...) \in \prod_{p+q=-1}$ is a cycle but not a boundary. If $d(x_0, x_1, x_2, ...) = (x_0 + 2x_1, x_1 + 2x_2, x_2 + 2x_3, ...) = (1, 1, 1, ...)$, then

$$\begin{gathered}
x_0 + 2x_1 = 1, \\
x_1 + 2x_2 = 1, \\
x_2 + 2x_3 = 1, \\
\vdots
\end{gathered}$$

By observation of parity, x_0 must be odd. Let $x_0 = 2n_0 + 1$. This forces $x_1 = -n_0$, for then

$$x_0 + 2x_1 = 2n_0 + 1 + 2(-n_0) = 2n_0 + 1 - 2n_0 = 1.$$

This causes $x_1 + 2x_2 = -n_0 + 2x_2$; for this to be equal to 1, we must have $n_0 = 2n_1 - 1$ itself odd and $x_2 = n_1$, for then

$$x_1 + 2x_2 = -n_0 + 2n_1 = -(2n_1 - 1) + 2n_1 = -2n_1 + 1 + 2n_1 = 1.$$

This causes $x_2 + 2x_3 = n_1 + 2x_3$; for this to be equal to 1, we must have $n_1 = 2n_2 + 1$ itself odd and $x_3 = -n_2$, for then

$$x_2 + 2x_3 = n_1 + 2(-n_2) = 2n_2 + 1 - 2n_2 = 1.$$

The pattern continues in this way: $x_i = (-1)^i n_{i-1}$ and $n_{i-1} = 2n_i + (-1)^i$.

We claim that for each $i \in \mathbf{N}$, $|x_{i+1}| < |x_i|$ when $|x_i| \neq 1$. Suppose this is not the case; i.e., $|x_i| \leq |x_{i+1}|$, and observe that

$$|x_i| \leq |x_{i+1}| = |n_i| = \left|\frac{n_{i-1} \pm 1}{2}\right| = \frac{1}{2}\left|n_{i-1} \pm 1\right| \leq \frac{1}{2}\left(|n_{i-1}| + 1\right) < |n_{i-1}| + 1 = |x_i| + 1.$$

Since $x_i, x_{i+1} \in \mathbf{Z}$ and $|x_i| \leq |x_{i+1}| < |x_i| + 1$, this forces $|x_i| = |x_{i+1}|$. But then we see that

$$\begin{aligned} & x_i + 2x_{i+1} = x_i + 2(\pm x_i) = x_i \pm 2x_i \\ & = \begin{cases} x_i + 2x_i = 3x_i, \text{ which is 1 when } x_i = \frac{1}{3}, \text{ a contradiction since } x_i \in \mathbf{Z}, \text{ or} \\ x_i - 2x_i = -x_i, \text{ which is 1 when } x_i = -1, \text{ a case we ruled out in our hypotheses.} \end{cases} \end{aligned}$$

Therefore $|x_{i+1}| < |x_i|$ when $|x_i| \neq 1$, as we claimed.

Since x_i is odd, after finitely many i, $|x_i| = 1$, and further, $x_i = 1$, since if $x_i = -1$, then

$$x_i + 2x_{i+1} = -1 + 2x_{i+1}$$

is equal to 1 when $x_{i+1} = 1$. So without loss of generality, $x_i = 1$. This forces a contradiction, as then

$$x_i + 2x_{i+1} = 1 + 2x_{i+1}$$

is 1 when $x_{i+1} = 0$, but then

$$x_{i+1} + 2x_{i+2} = 0 + 2x_{i+2} \neq 1.$$

Therefore, $(1, 1, 1, ...)$ is not in the image of d, and $\mathrm{Tot}^{\prod}(C)$ is not acyclic, as we wished to show.

Hom Cochain Complex 2.7.4 Given a chain complex P and a cochain complex I, form the double cochain complex $\mathrm{Hom}(P, I) = \{\mathrm{Hom}(P_p, I^q)\}$ using a variant of the sign trick. That is, if $f : P_p \to I^q$, then $d^h f : P_{p+1} \to I^q$ by $(d^h f)(p) = f(dp)$, while we define $d^v f : P_p \to I^{q+1}$ by

$$(d^v f)(p) = (-1)^{p+q+1} d(fp) \text{ for } p \in P_p.$$

$\mathrm{Hom}(P, I)$ is called the *Hom double complex*, and $\mathrm{Tot}^{\prod}(\mathrm{Hom}(P, I))$ is called the *(total) Hom cochain complex*. *Warning*: Different conventions abound in the literature. Bourbaki [BX] converts $\mathrm{Hom}(P, I)$ into a double chain complex and obtains a total Hom chain complex. Others convert I into a chain complex Q with $Q_q = I^{-q}$ and form $\mathrm{Hom}(P, Q)$ as a chain complex, and so on.

Morphisms and Hom 2.7.5 To explain our sign convention, suppose that C and D are two chain complexes. If we reindex D as a cochain complex, then an n-cycle f of $\mathrm{Hom}(C, D)$ is a sequence of maps $f_p : C_p \to D^{n-p} = D_{p-n}$ such that $f_p d = (-1)^n d f_{p+1}$, that is, a morphism of chain complexes from C to the translate $D[-n]$ of D. An n-boundary is a morphism f that is null homotopic. Thus $H^n \mathrm{Hom}(C, D)$ is the group of chain homotopy equivalence classes of morphisms $C \to D[-n]$, the morphisms in the quotient category $\mathbf{K}$ of the category of chain complexes discussed in exercise 1.4.5

Similarly, if X and Y are cochain complexes, we may form $\mathrm{Hom}(X, Y)$ by reindexing X. Our conventions about reindexing and translationg ensure that once again an n-cycle of $\mathrm{Hom}(X, Y)$ is a morphism $X \to Y[-n]$ and that $H^n \mathrm{Hom}(X, Y)$ is the group of chain homotopy equivalence classes of such morphisms. We will return to this point in Chapter 10 when we discuss $\mathbf{R}\mathrm{Hom}$ in the derived category $\mathbf{D}(\mathcal{A})$.

Exercise 2.7.3 To see why $\mathrm{Tot}^{\oplus}$ is used for the tensor product $P \otimes_R Q$ of right and left R-module complexes, while $\mathrm{Tot}^{\prod}$ is used for Hom, let I be a cochain complex of abelian groups. Show that there is a natural isomorphism of double complexes:

$$\mathrm{Hom}_{\mathbf{Ab}}(\mathrm{Tot}^{\oplus}(P \otimes_R Q), I) \cong \mathrm{Hom}_R(P, \mathrm{Tot}^{\prod}(\mathrm{Hom}_{\mathbf{Ab}}(Q, I))).$$

Recall that by definition, at degree n,

$$\mathrm{Hom}\left(\mathrm{Tot}^{\oplus}\left(P \otimes Q\right), I\right)^n = \mathrm{Hom}\left(\bigoplus_{r+s=p} P_r \otimes Q_s, I^q\right)^n.$$

for $p + q = n$. First, we claim the Hom functor preserves limits and colimits; i.e., if the limit $\lim X_i$ exists, then for all Y, $\mathrm{Hom}(Y, \lim X_i) \cong \lim \mathrm{Hom}(Y, X_i)$. Dually, if the colimit $\mathrm{colim}\, X_i$ exists, then for all Y, $\mathrm{Hom}(\mathrm{colim}\, X_i, Y) \cong \lim \mathrm{Hom}(X_i, Y)$. As coproducts are colimits and products are limits, this implies that

$$\operatorname{Hom}\left(\operatorname{Tot}^{\oplus}\left(P\otimes Q\right),I\right)^{n}=\operatorname{Hom}\left(\bigoplus_{r+s=p}P_r\otimes Q_s,I^q\right)^{n}\cong\left(\prod_{r+s=p}\operatorname{Hom}\left(P_r\otimes Q_s,I^q\right)\right)^{n}$$

naturally. By hom-tensor adjunction of modules, we have

$$\operatorname{Hom}\left(P_r\otimes Q_s,I^q\right)\cong\operatorname{Hom}\left(P_r,\operatorname{Hom}\left(Q_s,I^q\right)\right)$$

naturally. So set for each r

$$\begin{aligned}T(r)&=\prod_{r+s=p}\operatorname{Hom}(P_r\otimes Q_s,I^q)=\operatorname{Hom}(P_r\otimes Q_{p-r},I^q)\text{ and}\\ H(r)&=\prod_{r+s=p}\operatorname{Hom}(P_r,\operatorname{Hom}(Q_s,I^q))=\operatorname{Hom}(P_r,\operatorname{Hom}(Q_{p-r},I^q)).\end{aligned}$$

Note that $T(r_0)=\operatorname{Hom}(P_{r_0}\otimes Q_{p-r_0},I^q)\cong\operatorname{Hom}(P_{r_0},\operatorname{Hom}(Q_{p-r_0},I^q))=H(r_0)$ for any fixed r_0 by above, that this isomorphism extends to the product $\prod_{r\in\mathbf{Z}}T(r)\cong\prod_{r\in\mathbf{Z}}H(r)$, and that $\prod_{r\in\mathbf{Z}}T(r)\cong\operatorname{Hom}(\operatorname{Tot}^{\oplus}(P\otimes Q),I)^n$, since

$$\prod_{r\in\mathbf{Z}}\operatorname{Hom}(P_r\otimes Q_{p-r},I^q)\cong\operatorname{Hom}\left(\bigoplus_{r\in\mathbf{Z}}P_r\otimes Q_{p-r},I^q\right)=\operatorname{Hom}\left(\bigoplus_{r+s=p}P_r\otimes Q_s,I^q\right)$$

is the degree $n=p+q$ term of $\operatorname{Hom}(\operatorname{Tot}^{\oplus}(P\otimes Q),I)$. Next, for fixed n with $p+q=p'+q'=n$, we show that we have the natural isomorphism

$$\prod_{r\in\mathbf{Z}}H(r)=\prod_{r\in\mathbf{Z}}\operatorname{Hom}(P_r,\operatorname{Hom}(Q_{p-r},I^q))\cong\prod_{u+v=q'}\operatorname{Hom}(P_{p'},\operatorname{Hom}(Q_u,I^v)).$$

in degree n. To see this, let $r=p'$, let $u=p-r$ (so $p=r+u$), and let $v=q$. It follows that

$$\prod_{r\in\mathbf{Z}}\operatorname{Hom}(P_r,\operatorname{Hom}(Q_{p-r},I^q))\cong\prod_{p'\in\mathbf{Z}}\operatorname{Hom}(P_{p'},\operatorname{Hom}(Q_u,I^v)),$$

and since $n=p+q=r+u+v=p'+u+v=p'+q'$ is fixed, the product over p' is the same as the product over $u+v=q'$. Therefore, in degree $n=p+q=p'+q'$,

$$\begin{aligned}\operatorname{Hom}\left(\operatorname{Tot}^{\oplus}(P\otimes Q), I\right)^n \cong \prod_{r\in\mathbf{Z}} T(r) \cong \prod_{r\in\mathbf{Z}} H(r) &\cong \left(\prod_{u+v=q'} \operatorname{Hom}\left(P_{p'}, \operatorname{Hom}(Q_u, I^v)\right)\right)^n \\ &\cong \operatorname{Hom}\left(P_{p'}, \prod_{u+v=q'} \operatorname{Hom}(Q_u, I^v)\right)^n \\ &= \operatorname{Hom}\left(P, \operatorname{Tot}^{\prod}(Q, I)\right)^n.\end{aligned}$$

This holds for all n, so the isomorphism is shown.

•••

It only suffices now to prove the claim. We show that $\operatorname{Hom}(Y, \lim X_i) \cong \lim \operatorname{Hom}(Y, X_i)$; the dual is for free. A map γ in $\operatorname{Hom}(Y, \lim X_i)$ is uniquely determined by the following diagramatical definition of limit:

$$\begin{tikzcd} & Y \arrow[d, dashed, "\exists!\gamma"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\ & \lim X_i \arrow[dl] \arrow[dr] & \\ X_i \arrow[rr] & & X_j \end{tikzcd}$$

Now, $\lim \operatorname{Hom}(Y, X_i)$ is defined by the commutative diagram

$$\begin{tikzcd} & \lim \operatorname{Hom}(Y, X_i) \arrow[dl] \arrow[dr] & \\ \operatorname{Hom}(Y, X_i) \arrow[rr] & & \operatorname{Hom}(Y, X_j) \end{tikzcd}$$

In other words, $f \in \lim \operatorname{Hom}(Y, X_i)$ is an element determined by a collection of maps $\{Y \to X_i\}$, where a map $Y \to X_j$ is the same as a map $Y \to X_i \to X_j$. That is, f provides us the diagram

$$\begin{tikzcd} & Y \arrow[dl, bend right] \arrow[dr, bend left] & \\ X_i \arrow[rr] & & X_j \end{tikzcd}$$

which, by above, uniquely determines a map $\gamma \in \operatorname{Hom}(Y, \lim X_i)$. Conversely, a map $\gamma : Y \to \lim X_i$ gives us a collection of maps $\{Y \to X_i\}$ that respect the maps $X_i \to X_j$, and thus gives us an element f of $\lim \operatorname{Hom}(Y, X_i)$.

Theorem 2.7.6 *For every pair of R-modules A and B, and all n,*

$$\operatorname{Ext}^n_R(A,B) = R^n \operatorname{Hom}_R(A,-)(B) \cong R^n \operatorname{Hom}_R(-,B)(A).$$

Proof. Choose a projective resolution P of A and an injective resolution I of B. Form the first quadrant double cochain complex $\operatorname{Hom}(P,I)$. The augmentations induce maps from $\operatorname{Hom}(A,I)$ and $\operatorname{Hom}(P,B)$ to $\operatorname{Hom}(P,I)$. As in the proof of 2.7.2, the mapping cones of $\operatorname{Hom}(A,I) \to \operatorname{Tot}(\operatorname{Hom}(P,I))$ and $\operatorname{Hom}(P,B) \to \operatorname{Tot}(\operatorname{Hom}(P,I))$ are translates of the total complexes obtained from $\operatorname{Hom}(P,I)$ by adding $\operatorname{Hom}(A,I)[-1]$ and $\operatorname{Hom}(P,B)[-1]$, respectively. By the Acyclic Assembly Lemma 2.7.3 (or rather its dual), both mapping cones are exact. Therefore the maps

$$\operatorname{Hom}(A,I) \to \operatorname{Tot}(\operatorname{Hom}(P,I)) \leftarrow \operatorname{Hom}(P,B)$$

are quasi-isomorphisms. Taking cohomology yields the result:

$$\begin{aligned} R^* \operatorname{Hom}(A,-)(B) &= H^* \operatorname{Hom}(A,I) \\ &\cong H^* \operatorname{Tot}(\operatorname{Hom}(P,I)) \\ &\cong H^* \operatorname{Hom}(P,B) = R^* \operatorname{Hom}(-,B)(A). \end{aligned}$$

□

$$\begin{tikzcd}
\vdots & \vdots & \vdots & & \\
\operatorname{Hom}(A,I^2) \arrow[u] & \operatorname{Hom}(P_0,I^2) \arrow[u] \arrow[r] & \operatorname{Hom}(P_1,I^2) \arrow[u] \arrow[r] & \cdots & \\
\operatorname{Hom}(A,I^1) \arrow[u] & \operatorname{Hom}(P_0,I^1) \arrow[u] \arrow[r] & \operatorname{Hom}(P_1,I^1) \arrow[u] \arrow[r] & \operatorname{Hom}(P_2,I^1) \arrow[u] \arrow[r] & \cdots \\
\operatorname{Hom}(A,I^0) \arrow[u] & \operatorname{Hom}(P_0,I^0) \arrow[u] \arrow[r] & \operatorname{Hom}(P_1,I^0) \arrow[u] \arrow[r] & \operatorname{Hom}(P_2,I^0) \arrow[u] \arrow[r] & \cdots \\
 & \operatorname{Hom}(P_0,B) \arrow[r] & \operatorname{Hom}(P_1,B) \arrow[r] & \operatorname{Hom}(P_2,B) \arrow[r] & \cdots
\end{tikzcd}$$

Definition 2.7.7 ([CE]) In view of the two above theorems, the following definition seems natural. Let T be a left exact functor of p "variable" modules, some covariant and some contravariant. T will be called *right balanced* under the following conditions:

1. When any one of the covariant variables of T is replaced by an injective module, T becomes an exact functor in each of the remaining variables.
2. When any one of the contravariant variables of T is replaced by a projective module, T becomes an exact functor in each of the remaining variables. The functor Hom is an example of a right balanced functor, as is $\operatorname{Hom}(A \otimes B, C)$.

Exercise 2.7.4 Show that all p of the right derived functors $R^*T(A_1, \cdots, \widehat{A_i}, \cdots, A_p)(A_i)$ of T are naturally isomorphic.

Let T be a right balanced functor, without loss of generality with all covariant variables. Choose $i \in \{2,..,p\}$; we show that $R^*T(\widehat{A_1}, A_2, ..., A_p)(A_1) \cong R^*T(A_1, ..., \widehat{A_i}, ..., A_p)(A_i)$. Choose injective resolutions $A_1 \xrightarrow{\varepsilon} I^\bullet$ and $A_i \xrightarrow{\eta} J^\bullet$. Fix A_j for all $j \notin \{1, i\}$, and write $T(-,-) = T(-, A_2, ..., A_{i-1}, -, A_{i+1}, ..., A_p)$. Form the first quadrant double cochain complex:

$$
\begin{tikzcd}
\vdots & & \vdots & \vdots & \vdots & \\
T(A_1, J^2) \arrow[u] & & T(I^0, J^2) \arrow[u] \arrow[r, "d"] & T(I^1, J^2) \arrow[u] \arrow[r, "d"] & T(I^2, J^2) \arrow[u] \arrow[r] & \cdots \\
T(A_1, J^1) \arrow[u] & & T(I^0, J^1) \arrow[u, "d"] \arrow[r, "d"] & T(I^1, J^1) \arrow[u, "-d"] \arrow[r, "d"] & T(I^2, J^1) \arrow[u, "d"] \arrow[r] & \cdots \\
T(A_1, J^0) \arrow[u] & & T(I^0, J^0) \arrow[u, "d"] \arrow[r, "d"] & T(I^1, J^0) \arrow[u, "-d"] \arrow[r, "d"] & T(I^2, J^0) \arrow[u, "d"] \arrow[r] & \cdots \\
& & & & & \\
& & T(I^0, A_i) \arrow[r] & T(I^1, A_i) \arrow[r] & T(I^2, A_i) \arrow[r] & \cdots
\end{tikzcd}
$$

where the differentials in the horizontal direction are the images of $d : I^p \to I^{p+1}$ under the functor $T(-, B)$ where B is fixed, and the differentials in the vertical direction use the sign trick; i.e., they are $(-1)^p$ times the images of $d : J^q \to J^{q+1}$ under the functor $T(A, -)$ where A is fixed. Call the double complexes formed from I and J, from I and A_i, and from A_1 and J: $T(I, J)$, $T(I, A_i)$, and $T(A_1, J)$, respectively.

Now, consider the double complex C obtained from $T(I, J)$ by adding $T(A_1, J)[-1]$ in the column $p = -1$. By Exercise 1.2.8, the translate $\mathrm{Tot}(C)[+1]$ is the mapping cone of the map induced by ε from $\mathrm{Tot}(T(A_1, J)) = T(A_1, J)$ to $\mathrm{Tot}(T(I, J))$. By Corollary 1.5.4, that induced map $T(A_1, J) \to \mathrm{Tot}(T(I, J))$ is a quasi-isomorphism if and only if its mapping cone is exact; i.e., the complex $\mathrm{Tot}(C)[+1]$ is acyclic. The complex $\mathrm{Tot}(C)[+1]$ is acyclic if and only if $\mathrm{Tot}(C)$ is.

Since T is a right balanced functor, covariant in all variables, and $J^\bullet$ are injective modules, $T(-, J)$ is an exact functor. Thus, every row of C is exact, and as C is a right half-plane with exact rows, the (dual of the) Acyclic Assembly Lemma 2.7.3 implies that $\mathrm{Tot}(C)$ is acyclic, as desired. Thus the map induced by ε is a quasi-isomorphism; that is, $H^*(\mathrm{Tot}(T(I, J))) \cong H^*(T(A_1, J)) = R^*T(A_1, -)(A_i) = R^*T(A_1, ..., \widehat{A_i}, ..., A_p)(A_i)$.

Similarly, the double complex D obtained from $T(I, J)$ by adding $T(I, A_i)[-1]$ in the row $q = -1$ yields a totalization $\mathrm{Tot}(D)$. The translate $\mathrm{Tot}(D)[+1]$ is the mapping cone of the map induced by η from $\text{Tot}(T(I, A_i)) = T(I, A_i)$ to $\text{Tot}(T(I, J))$. We again show that $\text{Tot}(D)$ is acyclic to get the desired isomorphism on homology.

Indeed, as T is right balanced, covariant, and $I^\bullet$ are injective, $T(I, -)$ is exact, so every column of D is exact, and as D is a upper half-plane with exact columns, $\text{Tot}(D)$ is acyclic. Therefore, we may conclude $H^*(\text{Tot}(T(I, J))) \cong H^*(T(I, A_i)) = R^*T(-, A_i)(A_1) = R^*T(\widehat{A_1}, A_2, ..., A_p)(A_1)$.

By transitivity, $R^*T(A_1, ..., \widehat{A_i}, ..., A_p)(A_i) \cong R^*T(\widehat{A_1}, A_2, ..., A_p)(A_1)$, as we wished to show.

A similar discussion applies to right exact functors T which are *left balanced.* The prototype left balanced functor is $A \otimes B$. In particular, all of the left derived functors associated to a left balanced functor are isomorphic.

Application 2.7.8 (External product for Tor) Suppose that R is a commutative ring and that A, A', B, B' are R-modules. The *external product* is the map

$$\text{Tor}_i(A, B) \otimes_R \text{Tor}_j(A', B') \to \text{Tor}_{i+j}(A \otimes_R A', B \otimes_R B')$$

constructed for every i and j in the following manner. Choose projective resolutions $P \to A$, $P' \to A'$, and $P'' \to A \otimes A'$. The Comparison Theorem 2.2.6 gives a chain map $\text{Tot}(P \otimes P') \to P''$ which is unique up to chain homotopy equivalence. (We saw above that $H_i \text{Tot}(P \otimes P') = \text{Tor}_i(A, A')$, so we actually need the version of the Comparison Theorem contained in the porism 2.2.7.) This yields a natural map

$$H_n(P \otimes B \otimes P' \otimes B') \cong H_n(P \otimes P' \otimes B \otimes B') \to H_n(P'' \otimes B \otimes B') = \text{Tor}_n(A \otimes A', B \otimes B').$$

On the other hand, there are natural maps $H_i(C) \otimes H_j(C') \to H_{i+j} \text{Tot}(C \otimes C')$ for every pair of complexes C, C'; one maps the tensor product $c \otimes c'$ of cycles $c \in C_i$ and $c' \in {C'}_j$ to $c \otimes c' \in C_i \otimes {C'}_j$. (Check this!) The external product is obtained by composing the special case $C = P \otimes B$, $C' = P' \otimes B'$:

$$\text{Tor}_i(A, B) \otimes \text{Tor}_j(A', B') = H_i(P \otimes B) \otimes H_j(P' \otimes B') \to H_{i+j}(P \otimes B \otimes P' \otimes B')$$

with the above map.

Exercise 2.7.5

1. Show that the external product is independent of the choices of P, P', P'' and that it is natural in all four modules A, A', B, B'.
2. Show that the product is associative as a map to $\text{Tor}_*(A \otimes A' \otimes A'', B \otimes B' \otimes B'')$.
3. Show that the external product commutes with the connecting homomorphism δ in the long exact Tor sequences associated to $0 \to B_0 \to B \to B_1 \to 0$.
4. (Internal product) Suppose that A and B are R-algebras. Use (1) and (2) to show that $\text{Tor}_*^R(A, B)$ is a graded R-algebra.

There is an error in one of the maps describing the external product. The external product for Tor should be, for A, A', B, and B' R-modules and for $P_\bullet \to A$, ${P'}_\bullet \to A'$, and ${P''}_\bullet \to A \otimes A'$

projective resolutions, the following composition of maps and isomorphisms:

$$\mathrm{Tor}_i(A, B) \otimes \mathrm{Tor}_j(A', B') = H_i(P_\bullet \otimes B) \otimes H_j(P'_\bullet \otimes B') \to H_{i+j}(\mathrm{Tot}(P_\bullet \otimes B \otimes P'_\bullet \otimes B')).$$

Since tensor products of R-modules are commutative when R is a commutative ring, $B \otimes P'_n \cong P'_n \otimes B$ for all n, and thus $B \otimes P'_\bullet \cong P'_\bullet \otimes B$. Therefore

$$H_{i+j}(\mathrm{Tot}(P_\bullet \otimes B \otimes P'_\bullet \otimes B')) \cong H_{i+j}(\mathrm{Tot}(P_\bullet \otimes P'_\bullet \otimes B \otimes B')).$$

Since $B \otimes B'$ is only in degree 0, it commutes with the totalization, and we have

$$H_{i+j}(\mathrm{Tot}(P_\bullet \otimes P'_\bullet \otimes B \otimes B')) \cong H_{i+j}(\mathrm{Tot}(P_\bullet \otimes P'_\bullet) \otimes B \otimes B').$$

Since P_n and P'_n are projective, $P_n \otimes P'_n$ is projective for all n. Thus we have a projective chain complex (not necessarily a resolution) $\mathrm{Tot}(P_\bullet \otimes P'_\bullet) \to A \otimes A'$. We also have the projective resolution $P''_\bullet \to A \otimes A'$, so by Porism 2.2.7, the identity $\mathrm{id} : A \otimes A' \to A \otimes A'$ lifts to a unique, up to chain homotopy equivalence, chain map $\mathrm{Tot}(P_\bullet \otimes P'_\bullet) \to P''_\bullet$. After composing with the functor $(- \otimes B \otimes B')$ and homology, we have the map

$$H_{i+j}(\mathrm{Tot}(P_\bullet \otimes P'_\bullet) \otimes B \otimes B') \to H_{i+j}(P''_\bullet \otimes B \otimes B') = \mathrm{Tor}_{i+j}(A \otimes A', B \otimes B'),$$

and $\mathrm{Tor}_i(A, B) \otimes \mathrm{Tor}_j(A', B') \to \mathrm{Tor}_{i+j}(A \otimes A', B \otimes B')$ is defined to be the composition of the maps described.

1. Certainly the external product is independent of choice of P, P', and P''. Indeed, we know $\mathrm{Tor}_*(A, B) = H_*(P \otimes B)$, $\mathrm{Tor}_*(A', B') = H_*(P' \otimes B')$, and $\mathrm{Tor}_*(A \otimes A', B \otimes B') = H_*(P'' \otimes B \otimes B')$ are independent of choice of projective resolution, and the map $\mathrm{Tot}(P \otimes P') \to P''$ is unique up to chain homotopy equivalence, so its image under the functor $(- \otimes B \otimes B')$ is as well, and thus is well-defined on homology, which is to say, irrespective of P, P', and P''.

   To see naturality in A, A', B, and B', we need to show that the external product commutes with maps on its factors. That is, if $\varphi : A \to \widetilde{A}$, $\varphi' : A' \to \widetilde{A'}$, $\psi : B \to \widetilde{B}$, and $\psi' : B' \to \widetilde{B'}$ are maps, then the following diagram commutes.

$$
\begin{tikzcd}
\operatorname{Tor}_{i+j}(\widetilde{A} \otimes A', B \otimes B') & \operatorname{Tor}_{i+j}(A \otimes A', B \otimes B') \arrow[l] \arrow[r] & \operatorname{Tor}_{i+j}(A \otimes A', \widetilde{B} \otimes B') \\
\operatorname{Tor}_i(\widetilde{A}, B) \otimes \operatorname{Tor}_j(A', B') \arrow[u] & & \operatorname{Tor}_i(A, \widetilde{B}) \otimes \operatorname{Tor}_j(A', B') \arrow[u] \\
& \operatorname{Tor}_i(A, B) \otimes \operatorname{Tor}_j(A', B') \arrow[ul, bend right] \arrow[uu] \arrow[ur, bend left] \arrow[dl, bend left] \arrow[dd] \arrow[dr, bend right] & \\
\operatorname{Tor}_i(A, B) \otimes \operatorname{Tor}_j(\widetilde{A'}, B') \arrow[d] & & \operatorname{Tor}_i(A, B) \otimes \operatorname{Tor}_j(A', \widetilde{B'}) \arrow[d] \\
\operatorname{Tor}_{i+j}(A \otimes \widetilde{A'}, B \otimes B') & \operatorname{Tor}_{i+j}(A \otimes A', B \otimes B') \arrow[l] \arrow[r] & \operatorname{Tor}_{i+j}(A \otimes A', B \otimes \widetilde{B'})
\end{tikzcd}
$$

We show that one square commutes; all others will proceed similarly. Consider the square

$$
\begin{tikzcd}
\operatorname{Tor}_i(A, B) \otimes \operatorname{Tor}_j(A', B') \arrow[r] \arrow[d] & \operatorname{Tor}_i(\widetilde{A}, B) \otimes \operatorname{Tor}_j(A', B') \arrow[d] \\
\operatorname{Tor}_{i+j}(A \otimes A', B \otimes B') \arrow[r] & \operatorname{Tor}_{i+j}(\widetilde{A} \otimes A', B \otimes B')
\end{tikzcd}
$$

Let $P_\bullet \to A$, $P'_\bullet \to A'$, and $\widetilde{P}_\bullet \to \widetilde{A}$ be projective resolutions. Using the definition of Tor and the tensor-acyclicity of $P \otimes P'$ and $\widetilde{P} \otimes P'$, we must show

$$
\begin{tikzcd}
H_i(P \otimes B) \otimes H_j(P' \otimes B') \arrow[r] \arrow[d] & H_i(\widetilde{P} \otimes B) \otimes H_j(P' \otimes B') \arrow[d] \\
H_{i+j}(P \otimes P' \otimes B \otimes B') \arrow[r] & H_{i+j}(\widetilde{P} \otimes P' \otimes B \otimes B')
\end{tikzcd}
$$

The vertical maps are the natural maps $H_i(C) \otimes H_j(C') \to H_{i+j}(\operatorname{Tot}(C \otimes C'))$ composed with natural isomorphisms. Thus they commute with maps $P \to \widetilde{P}$ on homology. The map $P \to \widetilde{P}$ is uniquely determined on homology by the Comparison Theorem 2.2.6 applied to $\varphi : A \to \widetilde{A}$ and then composing with tensoring by $B \otimes B'$. Thus, by naturality, the square commutes, as desired.

2. Observe that

$$
\begin{aligned}
\big(\operatorname{Tor}_i(A, B) \otimes \operatorname{Tor}_j(A', B')\big) \otimes \operatorname{Tor}_k(A'', B'')& \\
&= \operatorname{Tor}_{i+j}(A \otimes A', B \otimes B') \otimes \operatorname{Tor}_k(A'', B'') \\
&= \operatorname{Tor}_{(i+j)+k}((A \otimes A') \otimes A'', (B \otimes B') \otimes B'')
\end{aligned}
$$

As addition is associative and tensor products are associative,

$$\begin{aligned}
\operatorname{Tor}_{(i+j)+k}((A \otimes A') \otimes A'', &(B \otimes B') \otimes B'') \\
&= \operatorname{Tor}_{i+(j+k)}(A \otimes (A' \otimes A''), B \otimes (B' \otimes B'')) \\
&= \operatorname{Tor}_i(A, B) \otimes \operatorname{Tor}_{j+k}(A' \otimes A'', B' \otimes B'') \\
&= \operatorname{Tor}_i(A, B) \otimes \big(\operatorname{Tor}_j(A', B') \otimes \operatorname{Tor}_k(A'', B'') \big).
\end{aligned}$$

Thus the external product is associative, as desired.

3. The connecting homomorphism δ is the map $\operatorname{Tor}_n(A, B_1) \to \operatorname{Tor}_{n-1}(A, B_0)$ in the long exact sequence

$$\begin{tikzcd}
 & & \ddots \arrow[dll, "\delta"] \\
\operatorname{Tor}_{n+1}(A, B_0) \arrow[r] & \operatorname{Tor}_{n+1}(A, B) \arrow[r] & \operatorname{Tor}_{n+1}(A, B_1) \arrow[dll, "\delta"] \\
\operatorname{Tor}_n(A, B_0) \arrow[r] & \operatorname{Tor}_n(A, B) \arrow[r] & \operatorname{Tor}_n(A, B_1) \arrow[dll, "\delta"] \\
\operatorname{Tor}_{n-1}(A, B_0) \arrow[r] & \operatorname{Tor}_{n-1}(A, B) \arrow[r] & \operatorname{Tor}_{n-1}(A, B_1) \arrow[dll, "\delta"] \\
\ddots & &
\end{tikzcd}$$

This is defined to be the map on homology, for $P_\bullet \to A$ a projective resolution:

$$\delta : H_n(P \otimes B_1) \to H_{n-1}(P \otimes B_0).$$

We must show that

$$\begin{tikzcd}
H_i(P \otimes B_1) \otimes H_j(P \otimes B') \arrow[r] \arrow[d] & H_{i+j}(P \otimes P \otimes B_1 \otimes B') \arrow[d] \\
H_{i-1}(P \otimes B_0) \otimes H_j(P \otimes B') \arrow[r] & H_{i+j-1}(P \otimes P \otimes B_0 \otimes B')
\end{tikzcd}$$

commutes. But since the snake lemma constructs a natural transformation, we have the commutative ladder

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\substack{H_{i+1}(P\otimes B_1)\\ \otimes \\ H_j(C')} \arrow[r] \arrow[d] & H_{i+j+1}(P \otimes B_1 \otimes C') \arrow[d] \\
\substack{H_{i}(P\otimes B_0)\\ \otimes \\ H_j(C')} \arrow[r] \arrow[d] & H_{i+j}(P \otimes B_0 \otimes C') \arrow[d] \\
\substack{H_{i}(P\otimes B)\\ \otimes \\ H_j(C')} \arrow[r] \arrow[d] & H_{i+j}(P \otimes B \otimes C') \arrow[d] \\
\substack{H_{i}(P\otimes B_1)\\ \otimes \\ H_j(C')} \arrow[r] \arrow[d] & H_{i+j}(P \otimes B_1 \otimes C') \arrow[d] \\
\substack{H_{i-1}(P\otimes B_0)\\ \otimes \\ H_j(C')} \arrow[r] \arrow[d] & H_{i+j-1}(P \otimes B_0 \otimes C') \arrow[d] \\
\vdots & \vdots
\end{tikzcd}
$$

which gives the desired commutativity.

4. An (associative) R algebra is a R-module V with a linear map $p : V \otimes V \to V$ satisfying the associative law. The grading means that V has a labeling on its elements by some monoid/group, and that the multiplication in the algebra is reflected in the multiplication in the monoid. And indeed, the underlying monoid is $(\mathbf{N}, +)$, for $\mathrm{Tor}_i(A, B) \otimes \mathrm{Tor}_j(A', B') = \mathrm{Tor}_{i+j}(A \otimes A', B \otimes B')$ is reflected in the grading $i + j \in \mathbf{N}$. Further, by part (2), the multiplication in the algebra is associative. Thus, $\mathrm{Tor}(A, B)$ is a graded R-algebra.

3.1 Tor for Abelian Groups

The first question many people ask about $\mathrm{Tor}_*(A,B)$ is "Why the name 'Tor'?" The results of this section should answer that question. Historically, the first Tor groups to arise were the groups $\mathrm{Tor}_1\left(\mathbf{Z}/p, B\right)$ associated to abelian groups. The following simple calculation describes these groups.

Calculation 3.1.1 $\mathrm{Tor}_0^{\mathbf{Z}}\left(\mathbf{Z}/p, B\right) = B/pB$, $\mathrm{Tor}_1^{\mathbf{Z}}\left(\mathbf{Z}/p, B\right) = {}_pB = \{b \in B \mid pb = 0\}$ and $\mathrm{Tor}_n^{\mathbf{Z}}\left(\mathbf{Z}/p, B\right) = 0$ for $n \geq 2$. To see this, use the resolution

$$0 \to \mathbf{Z} \xrightarrow{p} \mathbf{Z} \to \mathbf{Z}/p \to 0$$

to see that $\mathrm{Tor}_*\left(\mathbf{Z}/p, B\right)$ is the homology of the complex $0 \to B \xrightarrow{p} B \to 0$.

Proposition 3.1.2 *For all abelian groups A and B:*

(a) $\mathrm{Tor}_1^{\mathbf{Z}}(A,B)$ *is a torsion abelian group.*

(b) $\mathrm{Tor}_n^{\mathbf{Z}}(A,B) = 0$ *for $n \geq 2$.*

Proof. A is the direct limit of its finitely generated subgroups A_α, so by 2.6.17 $\mathrm{Tor}_n(A,B)$ is the direct limit of the $\mathrm{Tor}_n(A_\alpha, B)$. As the direct limit of torsion groups is a torsion group, we may assume that A is finitely generated, that is, $A \cong \mathbf{Z}^m \oplus \mathbf{Z}/p_1 \oplus \mathbf{Z}/p_2 \oplus \cdots \oplus \mathbf{Z}/p_r$ for appropriate integers $m, p_1, ..., p_r$. As $\mathbf{Z}^m$ is projective, $\mathrm{Tor}_n(\mathbf{Z}^m, -)$ vanishes for $n \neq 0$, and so we have

$$\mathrm{Tor}_n(A,B) \cong \mathrm{Tor}_n\left(\mathbf{Z}/p_1, B\right) \oplus \cdots \oplus \mathrm{Tor}_n\left(\mathbf{Z}/p_r, B\right).$$

The proposition holds in this case by calculation 3.1.1 above. $\square$

Proposition 3.1.3 $\mathrm{Tor}_1^{\mathbf{Z}}\left(\mathbf{Q}/\mathbf{Z}, B\right)$ *is the torsion subgroup of B for every abelian group B.*

Proof. As $\mathbf{Q}/\mathbf{Z}$ is the direct limit of its finite subgroups, each of which is isomorphic to $\mathbf{Z}/p$ for some integer p, and Tor commutes with direct limits,

$$\mathrm{Tor}_1^{\mathbf{Z}}\left(\mathbf{Q}/\mathbf{Z}, B\right) \cong \varinjlim \mathrm{Tor}_1^{\mathbf{Z}}\left(\mathbf{Z}/p, B\right) \cong \varinjlim ({}_pB) = \bigcup_p \{b \in B \mid pb = 0\},$$

which is the torsion subgroup of B. $\square$

Proposition 3.1.4 *If A is a torsionfree abelian group, then* $\mathrm{Tor}_n^{\mathbf{Z}}(A,B) = 0$ *for $n \neq 0$ and all abelian groups B.*

Proof. A is the direct limit of its finitely generated subgroups, each of which is isomorphic to $\mathbf{Z}^m$ for some m. Therefore, $\mathrm{Tor}_n(A,B) \cong \varinjlim \mathrm{Tor}_n(\mathbf{Z}^m, B) = 0$. $\square$

Remark (Balancing Tor) If R is any commutative ring, then $\mathrm{Tor}_*^R(A,B) \cong \mathrm{Tor}_*^R(B,A)$. In particular, this is true for $R = \mathbf{Z}$, that is, for abelian groups. This is because for fixed B, both are universal δ-functors over $F(A) = A \otimes B \cong B \otimes A$. Therefore $\mathrm{Tor}_1^{\mathbf{Z}}\left(A, \mathbf{Q}/\mathbf{Z}\right)$ is the torsion subgroup of A. From this we obtain the following.

Corollary 3.1.5 *For every abelian group A,*

$$\mathrm{Tor}_1^{\mathbf{Z}}(A,-) = 0 \iff A \text{ is torsionfree} \iff \mathrm{Tor}_1^{\mathbf{Z}}(-,A) = 0.$$

Calculation 3.1.6 All this fails if we replace $\mathbf{Z}$ with another ring. For example, if we take $R = \mathbf{Z}/m$ and $A = \mathbf{Z}/d$ with $d \mid m$, then we can use the periodic free resolution

$$\cdots \xrightarrow{d} \mathbf{Z}/m \xrightarrow{\frac{m}{d}} \mathbf{Z}/m \xrightarrow{d} \mathbf{Z}/m \xrightarrow{\varepsilon} \mathbf{Z}/d \to 0$$

to see that for all $\mathbf{Z}/m$-modules B we have

$$\operatorname{Tor}_n^{\mathbf{Z}/m}\left(\mathbf{Z}/d, B\right) = \begin{cases} B/dB & \text{if } n = 0 \\ \{b \in B \mid db = 0\}/\left(\frac{m}{d}\right)B & \text{if } n \text{ is odd, } n > 0 \\ \{b \in B \mid \left(\frac{m}{d}\right)b = 0\}/dB & \text{if } n \text{ is even, } n > 0. \end{cases}$$

Example 3.1.7 Suppose that $r \in R$ is a left nonzerodivisor on R, that is, ${}_rR = \{s \in R \mid rs = 0\}$ is zero. For every R-module B, set ${}_rB = \{b \in B \mid rb = 0\}$. We can repeat the above calculation with R/rR in place of $\mathbf{Z}/p$ to see that $\operatorname{Tor}_0\left(R/rR, B\right) = B/rB$, $\operatorname{Tor}_1^R\left(R/rR, B\right) = {}_rB$ and $\operatorname{Tor}_n^R\left(R/rR, B\right) = 0$ for all B when $n \geq 2$.

Exercise 3.1.1 If ${}_rR \neq 0$, all we have is the non-projective resolution

$$0 \to {}_rR \to R \xrightarrow{r} R \to R/rR \to 0.$$

Show that there is a short exact sequence

$$0 \to \operatorname{Tor}_2^R\left(R/rR, B\right) \to {}_rR \otimes_R B \xrightarrow{\text{multiply}} {}_rB \to \operatorname{Tor}_1^R\left(R/rR, B\right) \to 0$$

and that $\operatorname{Tor}_n^R\left(R/rR, B\right) \cong \operatorname{Tor}_{n-2}^R({}_rR, B)$ for $n \geq 3$.

Denote by m the multiply map ${}_rR \otimes B \to {}_rB$. Explicitly, $m : {}_rR \otimes B \to {}_rB$ is the map induced on the tensor product by ${}_rR \times B \to {}_rB$ defined by $(s, b) \mapsto sb$. As ${}_rR = \{s \in R \mid rs = 0\}$ and ${}_rB = \{b \in B \mid rb = 0\}$, m is well-defined, since $r(sb) = (rs)b = 0b = 0$ means $sb \in {}_rB$. Note that $\operatorname{im} m = \left\{\sum_k s_k b_k \mid (s_k, b_k) \in {}_rR \times B\right\} = {}_rRB$, since $\sum_k s_k b_k \in RB$ and

$$r\left(\sum_k s_k b_k\right) = \sum_k r(s_k b_k) = \sum_k (rs_k) b_k = \sum_k 0 b_k = \sum_k 0 = 0.$$

Thus, we have an exact sequence

$$0 \to \ker m \to {}_rR \otimes_R B \xrightarrow{m} {}_rB \to {}^{}_rB/{}_rRB \to 0.$$

The first part is shown if we can demonstrate that $\operatorname{Tor}_2^R\left(R/rR, B\right) \cong \ker m$ and that $\operatorname{Tor}_1^R\left(R/rR, B\right) \cong {}_rB/{}_rRB$.

To see that $\operatorname{Tor}_2^R\left(R/rR, B\right) \cong \ker m$, consider the short exact sequence $0 \to rR \to R \to R/rR \to 0$. This induces a long exact sequence of Tor modules:

$$
\begin{tikzcd}
 & & \ddots \arrow[dll, "\delta" description] & \\
\mathrm{Tor}_2^R(rR, B) \arrow[r] & \mathrm{Tor}_2^R(R, B) \arrow[r] & \mathrm{Tor}_2^R\left(R/_{rR}, B\right) \arrow[dll, "\delta" description] & \\
\mathrm{Tor}_1^R(rR, B) \arrow[r] & \mathrm{Tor}_1^R(R, B) \arrow[r] & \mathrm{Tor}_1^R\left(R/_{rR}, B\right) \arrow[dll, "\delta" description] & \\
rR \otimes B \arrow[r] & R \otimes B \arrow[r] & R/_{rR} \otimes B \arrow[r] & 0
\end{tikzcd}
$$

Note that $\mathrm{Tor}_n^R(R, B) = 0$ for $n \neq 0$, because R is free, hence projective, and has projective resolution $\cdots \to 0 \to 0 \to R$. If we then tensor by B, we get the complex $\cdots \to 0 \to 0 \to R \otimes B$, and computing homology gives $\mathrm{Tor}_n^R(R, B) = 0$ at every degree but 0. Thus the long exact sequence is

$$
\begin{tikzcd}
 & & \ddots \arrow[dll, "\delta" description] & \\
\mathrm{Tor}_2^R(rR, B) \arrow[r] & 0 \arrow[r] & \mathrm{Tor}_2^R\left(R/_{rR}, B\right) \arrow[dll, "\delta" description] & \\
\mathrm{Tor}_1^R(rR, B) \arrow[r] & 0 \arrow[r] & \mathrm{Tor}_1^R\left(R/_{rR}, B\right) \arrow[dll, "\delta" description] & \\
rR \otimes B \arrow[r] & R \otimes B \arrow[r] & R/_{rR} \otimes B \arrow[r] & 0
\end{tikzcd}
$$

so $\mathrm{Tor}_2^R\left(R/_{rR}, B\right) \cong \mathrm{Tor}_1^R(rR, B)$. Next, we claim the following short sequence is exact: $0 \to {}_rR \to R \to rR \to 0$. Indeed, ${}_rR \to R$ is injective since it's an inclusion, and $R \to rR$ is surjective since it's a projection. To see that $\ker(R \to rR) = \mathrm{im}({}_rR \to R)$, see that the kernel of $R \to rR$ is all elements $s \in R$ such that $rs = 0$, and $\mathrm{im}({}_rR \to R)$ is ${}_rR = \{s \in R \mid rs = 0\}$, so the claim holds.

Since the claimed sequence is a short exact sequence, we have another long exact sequence derived from Tor:

$$
\begin{tikzcd}
 & & \ddots \arrow[dll, "\delta"'] & \\
\mathrm{Tor}_1^R({}_rR,B) \arrow[r] & \mathrm{Tor}_1^R(R,B) \arrow[r] & \mathrm{Tor}_1^R(rR,B) \arrow[dll, "\delta"'] & \\
{}_rR\otimes B \arrow[r] & R\otimes B \arrow[r] & rR\otimes B \arrow[r] & 0
\end{tikzcd}
$$

and again, since $\mathrm{Tor}_1^R(R,B)=0$, we have

$$
\begin{tikzcd}
 & & \ddots \arrow[dll, "\delta"'] & \\
\mathrm{Tor}_1^R({}_rR,B) \arrow[r] & 0 \arrow[r] & \mathrm{Tor}_1^R(rR,B) \arrow[dll, "\delta"'] & \\
{}_rR\otimes B \arrow[r] & R\otimes B \arrow[r] & rR\otimes B \arrow[r] & 0
\end{tikzcd}
$$

so $\mathrm{Tor}_1^R(rR,B)\cong\ker({}_rR\otimes B\to R\otimes B)$. Call this map φ. Noting that $R\otimes_R B\cong B$ and $rR\otimes_R B\cong r(R\otimes_R B)\cong rB$, we thus have the commutative diagram with exact rows:

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}_1^R(rR,B) \arrow[r] & {}_rR\otimes B \arrow[r, "\varphi"] \arrow[d, "m"] & B \arrow[r] \arrow[d, "\mathrm{id}"] & rB \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
 & 0 \arrow[r] & {}_rB \arrow[r, "\iota"'] & B \arrow[r, "\pi"'] & rB \arrow[r] & 0
\end{tikzcd}
$$

By the exactness of the diagram, $\operatorname{im}\varphi=\ker\pi=\operatorname{im}\iota={}_rB$. Thus, we have the exact sequence

$$0\to\mathrm{Tor}_1^R(rR,B)\to{}_rR\otimes B\xrightarrow{\varphi}{}_rB\to 0,$$

$$0\to\mathrm{Tor}_1^R(rR,B)\to{}_rR\otimes B\xrightarrow{m}{}_rB\to 0,$$

and $\ker m\cong\ker\varphi\cong\mathrm{Tor}_1^R(rR,B)\cong\mathrm{Tor}_2^R\left(R/rR,B\right)$, as desired.

To see that $\mathrm{Tor}_1^R\left(R/rR,B\right)\cong{}_rB/{}_rRB$, return to the long exact sequence

$$\begin{tikzcd}
 & & \ddots \arrow[dll, rounded corners, to path={ -- ([yshift=2ex]\tikztostart.south) -| ([yshift=2ex]\tikztotarget.north) -- (\tikztotarget)}, "\delta"] & \\
\operatorname{Tor}_1^R(rR, B) \arrow[r] & 0 \arrow[r] & \operatorname{Tor}_1^R\left(R/rR, B\right) \arrow[dll, "\delta"] & \\
rR \otimes B \arrow[r] & R \otimes B \arrow[r] & R/rR \otimes B \arrow[r] & 0.
\end{tikzcd}$$

Here, $\operatorname{Tor}_1^R\left(R/rR, B\right) \cong \ker(rR \otimes B \to R \otimes B)$. Call this map ψ. Observe that we can compute $\ker\psi$; let $\sum_k rs_k \otimes b_k \in rR \otimes B$ be such that $\psi\left(\sum rs_k \otimes b_k\right) = 0$. As $R \otimes_R B \cong B$, $\psi\left(\sum rs_k \otimes b_k\right) = \sum rs_k b_k = r \sum s_k b_k$. This element is zero if and only if $\sum s_k b_k \in {}_rB$ by definition. Since $\sum rs_k \otimes b_k = r \sum s_k \otimes b_k = r \otimes \sum s_k b_k$, we see that $\ker\psi = \{r \otimes b \mid b \in {}_rB\}$. We have a map ${}_rB \to \ker\psi$ such that $b \mapsto r \otimes b$, and $r \otimes b = 0$ if and only if $b = \sum s_k b_k$ for some $s_k \in {}_rR$ and $b_k \in B$. Thus by the first isomorphism theorem, $\ker\psi \cong {}_rB/{}_rRB$, so $\operatorname{Tor}_1^R\left(R/rR, B\right) \cong {}_rB/{}_rRB$, as desired.

We proceed to the second part: that $\operatorname{Tor}_n^R\left(R/rR, B\right) \cong \operatorname{Tor}_{n-2}^R({}_rR, B)$ for $n \geq 3$. Returning once again to the long exact sequence

$$\begin{tikzcd}
 & & \ddots \arrow[dll, "\delta"] \\
\operatorname{Tor}_n^R(rR, B) \arrow[r] & 0 \arrow[r] & \operatorname{Tor}_n^R\left(R/rR, B\right) \arrow[dll, "\delta"] \\
\operatorname{Tor}_{n-1}^R(rR, B) \arrow[r] & 0 \arrow[r] & \operatorname{Tor}_{n-1}^R\left(R/rR, B\right) \arrow[dll, "\delta"] \\
\ddots & &
\end{tikzcd}$$

we see that $\operatorname{Tor}_n^R\left(R/rR, B\right) \cong \operatorname{Tor}_{n-1}^R(rR, B)$ for all $n \geq 2$. Returning to the second long exact sequence

$$\begin{tikzcd}
 & & \ddots \arrow[dll, "\delta"] \\
\mathrm{Tor}^R_{n-1}({}_rR, B) \arrow[r] & 0 \arrow[r] & \mathrm{Tor}^R_{n-1}(rR, B) \arrow[dll, "\delta"] \\
\mathrm{Tor}^R_{n-2}({}_rR, B) \arrow[r] & 0 \arrow[r] & \mathrm{Tor}^R_{n-2}(rR, B) \arrow[dll, "\delta"] \\
\ddots & &
\end{tikzcd}$$

we see that $\mathrm{Tor}^R_{n-1}(rR, B) \cong \mathrm{Tor}^R_{n-2}({}_rR, B)$ for all $n \geq 3$. Thus by transitivity, the result is shown.

Exercise 3.1.2 Suppose that R is a commutative domain with field of fractions F. Show that $\mathrm{Tor}^R_1\left(F/R, B\right)$ is the torsion submodule $\{b \in B \mid (\exists r \neq 0) rb = 0\}$ of B for every R-module B.

The short exact sequence $0 \to R \to F \to F/R \to 0$ gives rise to the long exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{Tor}^R_1(F, B) \arrow[r] & \mathrm{Tor}^R_1\left(F/R, B\right) \arrow[dll, "\delta"] \\
R \otimes_R B \arrow[r] & F \otimes_R B \arrow[r] & F/R \otimes_R B \arrow[r] & 0.
\end{tikzcd}$$

We claim a fraction field is always flat, so that $\mathrm{Tor}^R_n(F, B) = 0$ for $n \neq 0$, and specifically for $n = 1$. Using the fact that $R \otimes_R B \cong B$, this results in the exact sequence

$$\begin{tikzcd}
 & 0 \arrow[r] & \mathrm{Tor}^R_1\left(F/R, B\right) \arrow[dll, "\delta"] \\
B \arrow[r] & F \otimes B \arrow[r] & F/R \otimes B \arrow[r] & 0.
\end{tikzcd}$$

Hence, $\mathrm{Tor}^R_1\left(F/R, B\right) \cong \ker(B \to F \otimes B)$. Call this map φ. We now claim $\ker \varphi = \{b \in B \mid rb = 0 \text{ for some } r \neq 0\}$, and prove via double inclusion. Let b be such that $rb = 0$ for some $r \neq 0$. Compute $\varphi(b) = 1 \otimes b = \frac{1}{r} r \otimes b = \frac{1}{r} \otimes rb = \frac{1}{r} \otimes 0 = 0$, so $b \in \ker \varphi$. On the other hand, let $b \in \ker \varphi$, so $\varphi(b) = 1 \otimes b = 0$. Since $1 \otimes b$ is an elementary tensor, and an elementary tensor $x \otimes y$ is equal to 0 if and only if we can write it with either x or y equal to 0, and 1 is a unit in F, this means that $1 \otimes b = \frac{1}{r} \otimes rb = 0$ means there is some r such that $rb = 0$, and thus the double inclusion is shown. Therefore, $\operatorname{Tor}_1^R\left(F/R, B\right)$ is the torsion submodule of B, as desired.

Exercise 3.1.3 Show that $\operatorname{Tor}_1^R\left(R/I, R/J\right) \cong I \cap J/IJ$ for every right ideal I and left ideal J of R. In particular, $\operatorname{Tor}_1\left(R/I, R/I\right) \cong I/I^2$ for every 2-sided ideal I. *Hint*: Apply the Snake Lemma to

$$\begin{tikzcd}
0 \arrow[r] & IJ \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & I \otimes R/J \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J \arrow[r] & R \arrow[r] & R \otimes R/J \arrow[r] & 0.
\end{tikzcd}$$

Note that we may indeed apply the Snake Lemma; the top row is a short exact sequence, as $I \otimes R/J \cong I/IJ$, and the bottom row is a short exact sequence, as $R \otimes R/J \cong R/J$. The squares commute because the maps are

$$\begin{tikzcd}
IJ \arrow[r, "ij \mapsto ij"] \arrow[d, hook, "i_1"] & I \arrow[r, "i \mapsto i \otimes 1"] \arrow[d, hook, "i_2"] & I \otimes R/J \arrow[d, "i_2 \otimes \mathrm{id}"] \\
J \arrow[r, "j \mapsto j"] & R \arrow[r, "r \mapsto r \otimes 1"] & R \otimes R/J
\end{tikzcd}$$

So we apply the Snake Lemma, and get the long exact sequence

$$\begin{tikzcd}
 & & 0 \arrow[r] \arrow[d] & \ker(i_2 \otimes \mathrm{id}) \arrow[d] \arrow[dddll, bend left] & \\
 & IJ \arrow[r] \arrow[d] & I \arrow[r] \arrow[d, hook] & I \otimes R/J \arrow[r] \arrow[d, "i_2 \otimes \mathrm{id}"] & 0 \\
0 \arrow[r] & J \arrow[r] \arrow[d, dashed] & R \arrow[r] \arrow[d] & R \otimes R/J \arrow[r] \arrow[d] & 0 \\
 & J/IJ \arrow[r] & R/I \arrow[r] & \operatorname{coker}(i_2 \otimes \mathrm{id}) \arrow[r] & 0
\end{tikzcd}$$

meaning that $\ker(i_2 \otimes \mathrm{id}) \cong \ker\left(J/IJ \to R/I\right)$. However, we also have, from the short exact sequence $0 \to I \to R \to R/I \to 0$, the long exact sequence

$$\begin{tikzcd}
 & & \ddots \arrow[dll, "\delta"] \\
\mathrm{Tor}_1^R\left(I, R/J\right) \arrow[r] & \mathrm{Tor}_1^R\left(R, R/J\right) \arrow[r] & \mathrm{Tor}_1^R\left(R/I, R/J\right) \arrow[dll, "\delta"] \\
I \otimes R/J \arrow[r] & R \otimes R/J \arrow[r] & R/I \otimes R/J \arrow[r] & 0
\end{tikzcd}$$

As $\mathrm{Tor}_1^R(R, -) = 0$, we have

$$\begin{tikzcd}
 & 0 \arrow[r] & \mathrm{Tor}_1^R\left(R/I, R/J\right) \arrow[dll, "\delta"] \\
I \otimes R/J \arrow[r] & R \otimes R/J \arrow[r] & R/I \otimes R/J \arrow[r] & 0
\end{tikzcd}$$

so $\mathrm{Tor}_1^R\left(R/I, R/J\right) \cong \ker\left(I \otimes R/J \to R \otimes R/J\right) = \ker(i_2 \otimes \mathrm{id}) \cong \ker\left(J/IJ \to R/I\right)$. We claim that $\ker\left(J/IJ \to R/I\right) \cong I \cap J/IJ$; then the result is shown. Call the map φ so $\varphi(j + IJ) = j + I$. Indeed, the result is clear via double inclusion. If $j \in \ker \varphi$, then $j + IJ \in I$. Also, $j \in J$, but $j \notin IJ$. Thus $j \in I \cap J/IJ$. On the other hand, let $j \in I \cap J/IJ$. Computing $\varphi(j)$, we see that $\varphi(j) = j + I = 0 + I = 0$, so $j \in \ker \varphi$. Thus the result is shown.

3.2 Tor and Flatness

In the last chapter, we saw that if A is a right R-module and B is a left R-module, then $\mathrm{Tor}_*^R(A, B)$ may be computed either as the left derived functors of $A \otimes_R$ evaluated at B or as the left derived functors of $\otimes_R B$ evaluated at A. It follows that if either A or B is projective, then $\mathrm{Tor}_n(A, B) = 0$ for $n \neq 0$.

Definition 3.2.1 A left R-module B is *flat* if the functor $\otimes_R B$ is exact. Similarly, a right R-module A is *flat* if the functor $A \otimes_R$ is exact. The above remarks show that projective modules are flat. The example $R = \mathbf{Z}$, $B = \mathbf{Q}$ shows that flat modules need not be projective.

Theorem 3.2.2 *If S is a central multiplicatively closed set in a ring R, then $S^{-1}R$ is a flat R-module.*

Proof. Form the filtered category I whose objects are the elements of S and whose morphisms are $\mathrm{Hom}_I(s_1, s_2) = \{s \in S \mid s_1 s = s_2\}$. Then $\underrightarrow{\mathrm{colim}}\, F(s) \cong S^{-1}R$ for the functor $F : I \to R$-**mod** defined by $F(s) = R$, $F(s_1 \xrightarrow{s} s_2)$ being multiplication by s. (*Exercise*: Show that the maps $F(s) \to S^{-1}R$ sending 1 to $\frac{1}{s}$ induce an isomorphism $\underrightarrow{\mathrm{colim}}\, F(s) \cong S^{-1}R$.) Since $S^{-1}R$ is the filtered colimit of the free R-modules $F(s)$, it is flat by 2.6.17. $\square$

Exercise 3.2.1 Show that the following are equivalent for every left R-module B.

1. B is flat.
2. $\mathrm{Tor}_n^R(A,B)=0$ for all $n\neq 0$ and all A.
3. $\mathrm{Tor}_1^R(A,B)=0$ for all A.

We show that

$$\begin{tikzcd}
& 1. \arrow[dr, Rightarrow] & \\
3. \arrow[ur, Rightarrow] & & 2. \arrow[ll, Rightarrow]
\end{tikzcd}$$

For 1. implies 2., let B be flat, so by definition, $\otimes_R B$ is exact. Let A be any R-module, and choose a projective resolution $P_\bullet \to A$. As $P_\bullet$ is a resolution, it is exact except at P_0, and as $-\otimes_R B$ is exact, $P_\bullet \otimes_R B$ is exact except at $P_0\otimes_R B$. Thus $\mathrm{Tor}_n^R(A,B)=H_n(P_\bullet\otimes B)=0$ when $n\neq 0$.

For 2. implies 3., if $\mathrm{Tor}_n^R(A,B)=0$ for all $n\neq 0$ and all A, then certainly for $n=1$.

For 3. implies 1., let $\mathrm{Tor}_1^R(A,B)=0$ for all A. Let $0\to L\to M\to N\to 0$ be an arbitrary short exact sequence. The resulting Tor long exact sequence is

$$\begin{tikzcd}
& \cdots \arrow[r] & \mathrm{Tor}_1(N,B) \arrow[dll, "\delta"] & \\
L\otimes B \arrow[r] & M\otimes B \arrow[r] & N\otimes B \arrow[r] & 0.
\end{tikzcd}$$

As $\mathrm{Tor}_1(N,B)=0$ by hypothesis, we have the short exact sequence

$$0\to L\otimes B\to M\otimes B\to N\otimes B\to 0,$$

so $\otimes B$ is exact, and B is flat, as we wished to show.

Exercise 3.2.2 Show that if $0\to A\to B\to C\to 0$ is exact and both B and C are flat, then A is also flat.

Let $0\to A\to B\to C\to 0$ be exact and B and C flat. Let D be any module. The resulting Tor long exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{Tor}_2(C,D) \arrow[dll, "\delta"] \\
\mathrm{Tor}_1(A,D) \arrow[r] & \mathrm{Tor}_1(B,D) \arrow[r] & \mathrm{Tor}_1(C,D) \arrow[dll, "\delta"] \\
A \otimes D \arrow[r] & B \otimes D \arrow[r] & C \otimes D \arrow[r] & 0
\end{tikzcd}$$

simplifies, since B and C are flat, to

$$\begin{tikzcd}
 & \cdots \arrow[r] & 0 \arrow[dll, "\delta"] \\
\mathrm{Tor}_1(A,D) \arrow[r] & 0 \arrow[r] & 0 \arrow[dll, "\delta"] \\
A \otimes D \arrow[r] & B \otimes D \arrow[r] & C \otimes D \arrow[r] & 0,
\end{tikzcd}$$

by Exercise 3.2.1. By exactness, $\mathrm{Tor}_1(A, D) = 0$ for any D, and again by Exercise 3.2.1, A is flat, as desired.

Exercise 3.2.3 We saw in the last section that if $R = \mathbf{Z}$ (or more generally, if R is a principal ideal domain), a module B is flat iff B is torsionfree. Here is an example of a torsionfree ideal I that is not a flat R-module. Let k be a field and set $R = k[x, y]$, $I = (x, y)R$. Show that $k = R/I$ has the projective resolution

$$0 \to R \xrightarrow{\begin{bmatrix} -y \\ x \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} x & y \end{bmatrix}} R \to k \to 0.$$

Then compute that $\mathrm{Tor}_1^R(I, k) \cong \mathrm{Tor}_2^R(k, k) \cong k$, showing that I is not flat.

This is a free resolution, not merely projective. See that $k[x, y]$ surjects onto $k[x, y]/(x, y)$ via $f \overset{\varepsilon}{\mapsto} [f]$. It has kernel (x, y). As there are two generators of the ideal (x, y), we have the map $d_1 : k[x, y]^2 \to k[x, y]$ given by $d_1 = \begin{bmatrix} x & y \end{bmatrix}$. The kernel of d_1 is generated by $\begin{bmatrix} -y \\ x \end{bmatrix}$, since

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = fx + gy,$$

and this is zero precisely when described. Thus, the free resolution is

$$0 \to k[x, y] \xrightarrow{\begin{bmatrix} -y \\ x \end{bmatrix}} k[x, y]^2 \xrightarrow{\begin{bmatrix} x & y \end{bmatrix}} k[x, y] \to k[x, y]/(x, y) \to 0,$$

as provided. Using this resolution over k to compute $\mathrm{Tor}_2^R(k,k)$, we have the complex

$$0 \to R \otimes_R k \xrightarrow{\left[\begin{smallmatrix}-y\\x\end{smallmatrix}\right]\otimes \mathrm{id}_k} R^2 \otimes_R k \xrightarrow{[\,x\ y\,]\otimes \mathrm{id}_k} R \otimes_R k \to 0.$$

Now, note that $R\otimes_R k \cong k$, and that $R^2\otimes_R k = (R\oplus R)\otimes_R k \cong (R\otimes_R k)\oplus(R\otimes_R k) \cong k\oplus k = k^2$. By defintion, $\mathrm{Tor}_2^R(k,k)$ is the second homology of the complex above, which is

$$\ker\left(\left[\begin{smallmatrix}-y\\x\end{smallmatrix}\right] \otimes \mathrm{id}_k\right)\Big/\mathrm{im}(0 \to R) = \ker\left(\left[\begin{smallmatrix}-y\\x\end{smallmatrix}\right] \otimes \mathrm{id}_k\right).$$

The map $\left[\begin{smallmatrix}-y\\x\end{smallmatrix}\right] : R\otimes k \to R^2 \otimes k$ corresponds to the map $\left[\begin{smallmatrix}-y\\x\end{smallmatrix}\right] : k \to k^2$ under the isomorphisms above. Examining this map, we find that

$$\left[\begin{smallmatrix}-y\\x\end{smallmatrix}\right]\left[\,f\,\right] = \begin{bmatrix}-fy\\fx\end{bmatrix} \in k^2 \cong \left(R\big/I\right)^2 = \left(R\big/(x,y)\right)^2,$$

so $-fy = fx = 0$ in $R\big/(x,y)$, and thus the kernel of $\left[\begin{smallmatrix}-y\\x\end{smallmatrix}\right] : k \to k^2$ is all of k. Hence, $\mathrm{Tor}_2^R(k,k) \cong k$, as claimed.

The problem is complete once we show $\mathrm{Tor}_1^R(I,k) \cong \mathrm{Tor}_2^R(k,k)$. To that end, we have the short exact sequence

$$0 \to I \to R \to R\big/I \to 0$$
$$0 \to I \to R \to k \to 0$$

which gives rise to the long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{Tor}_2^R(R,k) \arrow[r] & \mathrm{Tor}_2^R(k,k) \arrow[dll, "\delta"] \\ \mathrm{Tor}_1^R(I,k) \arrow[r] & \mathrm{Tor}_1^R(R,k) \arrow[r] & \cdots \end{tikzcd}$$

As R is free, hence flat, $\mathrm{Tor}_n^R(R,k) = 0$ for $n \neq 0$ in general and $n = 1,2$ in particular. Hence

$$\begin{tikzcd} & 0 \arrow[r] & \mathrm{Tor}_2^R(k,k) \arrow[dll, "\delta"] \\ \mathrm{Tor}_1^R(I,k) \arrow[r] & 0, & \end{tikzcd}$$

so $\mathrm{Tor}_2(k,k) \cong \mathrm{Tor}_1(I,k)$, as desired.

Definition 3.2.3 The *Pontrjagin dual* B^* of a left R-module B is the right R-module $\mathrm{Hom}_{\mathbf{Ab}}\left(B, {}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}\right)$; an element r of R acts via $(fr)(b) = f(rb)$.

Proposition 3.2.4 *The following are equivalent for every left R-module B:*

1. *B is a flat R-module.*
2. *B^* is an injective right R-module.*
3. *$I \otimes_R B \cong IB = \{x_1b_1 + \cdots + x_nb_n \in B \mid x_i \in I, b_i \in B\} \subseteq B$ for every right ideal I of R.*
4. *$\mathrm{Tor}_1^R\left({}^{R}\!/\!_{I}, B\right) = 0$ for every right ideal I of R.*

Proof. The equivalence of (3) and (4) follows from the exact sequence

$$0 \to \mathrm{Tor}_1\left({}^{R}\!/\!_{I}, B\right) \to I \otimes B \to B \to {}^{B}\!/\!_{IB} \to 0.$$

Now for every inclusion $A' \subseteq A$ of right modules, the adjoint functors $\otimes B$ and $\mathrm{Hom}(-, B)$ give a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(A, B^*) & \longrightarrow & \mathrm{Hom}(A', B^*) \\ \downarrow\cong & & \downarrow\cong \\ (A \otimes B)^* = \mathrm{Hom}\left(A \otimes B, {}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}\right) & \longrightarrow & \mathrm{Hom}\left(A' \otimes B, {}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}\right) = (A' \otimes B)^*. \end{array}$$

Using the lemma below and Baer's criterion 2.3.1, we see that

$$\begin{aligned} B^* \text{ is injective } &\iff (A \otimes B)^* \to (A' \otimes B)^* \text{ is surjective for all } A' \subseteq A. \\ &\iff A' \otimes B \to A \otimes B \text{ is injective for all } A' \subseteq A \iff B \text{ is flat.} \\ B^* \text{ is injective } &\iff (R \otimes B)^* \to (I \otimes B)^* \text{ is surjective for all } I \subseteq R \\ &\iff I \otimes B \to R \otimes B \text{ is injective for all } I \\ &\iff I \otimes B \cong IB \text{ for all } I. \end{aligned}$$

□

Lemma 3.2.5 *A map $f : B \to C$ is injective iff the dual map $f^* : C^* \to B^*$ is surjective.*

Proof. If A is the kernel of f, then A^* is the cokernel of f^*, because $\mathrm{Hom}\left(-, {}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}\right)$ is contravariant exact. But we saw in exercise 2.3.3 that $A = 0$ iff $A^* = 0$. □

Exercise 3.2.4 Show that a sequence $A \to B \to C$ is exact iff its dual $C^* \to B^* \to A^*$ is exact.

The group ${}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}$ is divisible, which in $\mathbf{Ab}$ is the case if and only if it is injective, by Corollary 2.3.2. By Lemma 2.3.4, ${}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}$ is injective in $\mathbf{Ab}$ if and only if the contravariant functor $\mathrm{Hom}_{\mathbf{Ab}}\left(-, {}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}\right)$ is exact. Yet, the Pontrjagin dual is a functor from R-mod. To remedy this, note that the forgetful functor Forget : R-mod $\to \mathbf{Ab}$ is exact, so $\mathrm{Hom}_{\mathbf{Ab}}\left(\mathrm{Forget}(-), {}^{\mathbf{Q}}\!/\!_{\mathbf{Z}}\right) = (-)^*$, the functor taking the Pontrjagin dual, is exact, and thus for an exact sequence $A \to B \to C$, the sequence $C^* \to B^* \to A^*$ is exact, as desired.

To prove the other direction, it suffices to address the only claim in the proof above that wasn't explicitly if and only if; namely, that for Forget : R-mod $\to \mathbf{Ab}$, a sequence $A \to B \to C$ is exact if and only if $\mathrm{Forget}(A) \to \mathrm{Forget}(B) \to \mathrm{Forget}(C)$ is exact. This is clearly the case; $\mathrm{im}(A \to B) = \ker(B \to C)$ in R-mod if and only if $\mathrm{im}(A \to B) = \ker(B \to C)$ in **Ab**, since images and kernels are closed under scalar multiplication, so whether the structure is regarded or not does not affect the image and kernel.

An R-module M is called *finitely presented* if it can be presented using finitely many generators $(e_1, ..., e_n)$ and relations $(\sum \alpha_{ij} e_j = 0,\ i = 1, ..., m)$. That is, there is an $m \times n$ matrix α and an exact sequence $R^m \xrightarrow{\alpha} R^n \to M \to 0$. If M is finitely generated, the following exercise shows that the property of being finitely presented is independent of the choice of generators.

Exercise 3.2.5 Suppose that $\varphi : F \to M$ is *any* surjection, where F is finitely generated and M is finitely presented. Use the Snake Lemma to show that $\ker(\varphi)$ is finitely generated.

Let's set up the diagram necessary to apply the Snake Lemma. We have the exact sequence $R^m \xrightarrow{\alpha} R^n \xrightarrow{\psi} M \to 0$, since M is finitely presented, and we have the short exact sequence $0 \to \ker\varphi \hookrightarrow F \xrightarrow{\varphi} M \to 0$ since φ is a surjection. We therefore have the desired rows, and one vertical map:

$$\begin{tikzcd}
 & R^m \arrow[r, "\alpha"] & R^n \arrow[r, "\psi"] & M \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \ker\varphi \arrow[r] & F \arrow[r, "\varphi"] & M \arrow[r] & 0
\end{tikzcd}$$

As R^n is a free module, it is projective, so by definition of projective, there exists a map f such that the following diagram commutes:

$$\begin{tikzcd}
 & R^n \arrow[dl, dashed, "f"'] \arrow[d, "\mathrm{id}\,\psi"] & \\
F \arrow[r, "\varphi"'] & M \arrow[r] & 0
\end{tikzcd}$$

Thus the second vertical map exists such that the right square commutes.

$$\begin{tikzcd}
 & R^m \arrow[r, "\alpha"] & R^n \arrow[r, "\psi"] \arrow[d, "f"] & M \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \ker\varphi \arrow[r] & F \arrow[r, "\varphi"] & M \arrow[r] & 0
\end{tikzcd}$$

Finally, we claim that $\varphi f \alpha = 0$, so that $f\alpha \in \ker\varphi$, giving us the final vertical map. Indeed, by the commutativity of the square, $\varphi f \alpha = \mathrm{id}\,\psi\alpha = \psi\alpha$, which is 0 by exactness of the top row. Hence we have the requisite Snake Lemma diagram:

$$
\begin{tikzcd}
 & \ker f\alpha \arrow[r] \arrow[d] & \ker f \arrow[r] \arrow[d] & 0 \arrow[d, dashed] \arrow[dddll, out=0, in=180, looseness=1.5] & \\
 & R^m \arrow[r, "\alpha"] \arrow[d, "f\alpha"] & R^n \arrow[r] \arrow[d, "f"] & M \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \ker\varphi \arrow[r] \arrow[d] & F \arrow[r, "\varphi"] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & \ker\varphi / \operatorname{im} f\alpha \arrow[r] & F/\operatorname{im} f \arrow[r] & 0 &
\end{tikzcd}
$$

Hence, by the exactness of the Snake Lemma long exact sequence, $\ker\varphi/\operatorname{im} f\alpha \cong F/\operatorname{im} f$. Now, F is finitely generated by hypothesis, $\operatorname{im} f$ is finitely generated since f surjects onto its image (i.e., $R^n \xrightarrow{f} \operatorname{im} f \to 0$), and $F/\operatorname{im} f$ is finitely generated since $R^n \xrightarrow{f} F$ and $F \to F/\operatorname{im} f$ are both surjections and hence their composition $R^n \to F/\operatorname{im} f$ is as well. By the isomorphism, $\ker\varphi/\operatorname{im} f\alpha$ is finitely generated. Note also that $\operatorname{im} f\alpha$ is finitely generated, for the same reason $\operatorname{im} f$ was. Thus we have a diagram with exact rows and column:

$$
\begin{tikzcd}
 & 0 \arrow[d] & \\
R^k \arrow[r] & \operatorname{im} f\alpha \arrow[r] \arrow[d] & 0 \\
 & \ker\varphi \arrow[d] & \\
R^\ell \arrow[r] & \ker\varphi/\operatorname{im} f\alpha \arrow[r] \arrow[d] & 0 \\
 & 0 &
\end{tikzcd}
$$

As R^k and R^ℓ are free, they are projective, so we may apply the Horseshoe Lemma 2.2.8 to get

$$
\begin{tikzcd}
 & 0 \arrow[d] & \\
R^k \arrow[r] & \operatorname{im} f\alpha \arrow[r] \arrow[d] & 0 \\
R^k \oplus R^\ell = R^{k+\ell} \arrow[r] & \ker\varphi \arrow[r] \arrow[d] & 0 \\
R^\ell \arrow[r] & \ker\varphi/\operatorname{im} f\alpha \arrow[r] \arrow[d] & 0 \\
 & 0 &
\end{tikzcd}
$$

Thus, $\ker\varphi$ is finitely generated, as desired.

Still letting A^* denote the Pontrjagin dual 3.2.3 of A, there is a natural map $\sigma : A^* \otimes_R M \to \operatorname{Hom}_R(M, A)^*$

defined by $\sigma(f \otimes m) : h \mapsto f(h(m))$ for $f \in A^*$, $m \in M$, and $h \in \mathrm{Hom}(M, A)$. (*Exercise*: If $M = \oplus_{i=1}^{\infty} R$, show that σ is not an isomorphism.)

Lemma 3.2.6 *The map σ is an isomorphism for every finitely presented M and all A.*

Proof. A simple calculation shows that σ is an isomorphism if $M = R$. By additivity, σ is an isomorphism if $M = R^m$ or R^n. Now consider the diagram

$$\begin{array}{ccccccc}
A^* \otimes R^m & \longrightarrow & A^* \otimes R^n & \longrightarrow & A^* \otimes M & \longrightarrow & 0 \\
\sigma \downarrow \cong & & \sigma \downarrow \cong & & \sigma \downarrow & & \\
\mathrm{Hom}(R^m, A)^* & \xrightarrow{\alpha^*} & \mathrm{Hom}(R^n, A)^* & \longrightarrow & \mathrm{Hom}(M, A)^* & \longrightarrow & 0.
\end{array}$$

The rows are exact because $\otimes$ is right exact, Hom is left exact, and Pontrjagin dual is exact by 2.3.3. The 5-lemma shows that σ is an isomorphism. $\square$

Theorem 3.2.7 *Every finitely presented flat R-module M is projective.*

Proof. In order to show that M is projective, we shall show that $\mathrm{Hom}_R(M, -)$ is exact. To this end, suppose that we are given a surjection $B \to C$. Then $C^* \to B^*$ is an injection, so if M is flat, the top arrow of the square

$$\begin{array}{ccc}
(C^*) \otimes_R M & \longrightarrow & (B^*) \otimes_R M \\
\cong \downarrow & & \cong \downarrow \\
\mathrm{Hom}(M, C)^* & \longrightarrow & \mathrm{Hom}(M, B)^*
\end{array}$$

is an injection. Hence the bottom arrow is an injection. As we have seen, this implies that $\mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C)$ is a surjection, as required. $\square$

Flat Resolution Lemma 3.2.8 *The groups $\mathrm{Tor}_*(A, B)$ may be computed using resolutions by flat modules. That is, if $F \to A$ is a resolution of A with the F_n being flat modules, then $\mathrm{Tor}_*(A, B) \cong H_*(F \otimes B)$. Similarly, if $F' \to B$ is a resolution of B by flat modules, then $\mathrm{Tor}_*(A, B) \cong H_*(A \otimes F')$.*

Proof. We use induction and dimension shifting (exercise 2.4.3) to prove that $\mathrm{Tor}_n(A, B) \cong H_n(F \otimes B)$ for all n; the second part follows by arguing over R^{op}. The assertion is true for $n = 0$ because $\otimes B$ is right exact. Let K be such that $0 \to K \to F_0 \to A \to 0$ is exact; if $E = (\cdots \to F_2 \to F_1 \to 0)$, then $E \to K$ is a resolution of K by flat modules. For $n = 1$ we simply compute

$$\begin{aligned}
\mathrm{Tor}_1(A, B) &= \ker(K \otimes B \to F_0 \otimes B) \\
&= \ker \left\{ {F_1 \otimes B}/{\mathrm{im}(F_2 \otimes B)} \to F_0 \otimes B \right\} = H_1(F \otimes B).
\end{aligned}$$

For $n \geq 2$ we use induction to see that

$$\mathrm{Tor}_n(A, B) \cong \mathrm{Tor}_{n-1}(K, B) \cong H_{n-1}(E \otimes B) = H_n(F \otimes B).$$

$\square$

Proposition 3.2.9 (Flat base change for Tor) *Suppose $R \to T$ is a ring map such that T is flat as an R-module. Then for all R-modules A, all T-modules C and all n*

$$\mathrm{Tor}_n^R(A, C) \cong \mathrm{Tor}_n^T(A \otimes_R T, C).$$

Proof. Choose an R-module projective resolution $P \to A$. Then $\mathrm{Tor}_*^R(A, C)$ is the homology of $P \otimes_R C$. Since T is R-flat, and each $P_n \otimes_R T$ is a projective T-module, $P \otimes T \to A \otimes T$ is a T-module projective *resolution.* Thus $\mathrm{Tor}_*^T(A \otimes_R T, C)$ is the homology of the complex $(P \otimes_R T) \otimes_T C \cong P \otimes_R C$ as well. $\square$

Corollary 3.2.10 *If R is commutative and T is a flat R-algebra, then for all R-modules A and B, and for all n*

$$T \otimes_R \operatorname{Tor}_n^R(A, B) \cong \operatorname{Tor}_n^T(A \otimes_R T, T \otimes_R B).$$

Proof. Setting $C = T \otimes_R B$, it is enough to show that $\operatorname{Tor}_*^R(A, T \otimes B) = T \otimes \operatorname{Tor}_*^R(A, B)$. As $T \otimes_R$ is an exact functor, $T \otimes \operatorname{Tor}_*^R(A, B)$ is the homology of $T \otimes_R (P \otimes_R B) \cong P \otimes_R (T \otimes_R B)$, the complex whose homology is $\operatorname{Tor}_*^R(A, T \otimes B)$. $\square$

Now we shall suppose that R is a commutative ring, so that the $\operatorname{Tor}_*^R(A, B)$ are actually R-modules in order to show how Tor_* localizes.

Lemma 3.2.11 *If $\mu : A \to A$ is multiplication by a central element $r \in R$, so are the induced maps $\mu_* : \operatorname{Tor}_n^R(A, B) \to \operatorname{Tor}_n^R(A, B)$ for all n and B.*

Proof. Pick a projective resolution $P \to A$. Multiplication by r is an R-module chain map $\widetilde{\mu} : P \to P$ over μ (this uses the fact that r is central), and $\widetilde{\mu} \otimes B$ is multiplication by r on $P \otimes B$. The induced map μ_* on the subquotient $\operatorname{Tor}_n(A, B)$ of $P_n \otimes B$ is therefore also multiplication by r. $\square$

Corollary 3.2.12 *If A is an ${}^{R}\!/_{r}$-module, then for every R-module B the R-modules $\operatorname{Tor}_*^R(A, B)$ are actually ${}^{R}\!/_{r}$-modules, that is, annihilated by the ideal rR.*

Corollary 3.2.13 (Localization for Tor) *If R is commutative and A and B are R-modules, then the following are equivalent for each n:*

1. $\operatorname{Tor}_n^R(A, B) = 0$.
2. *For every prime ideal p of R* $\operatorname{Tor}_n^{R_p}(A_p, B_p) = 0$.
3. *For every maximal ideal m of R* $\operatorname{Tor}_n^{R_m}(A_m, B_m) = 0$.

Proof. For any R-module M, $M = 0 \iff M_p = 0$ for every prime $p \iff M_m = 0$ for every maximal ideal m. In the case $M = \operatorname{Tor}_n^R(A, B)$ we have

$$M_p = R_p \otimes_R M = \operatorname{Tor}_n^{R_p}(A_p, B_p).$$

$\square$

3.3 Ext for Nice Rings

We first turn to a calculation of $\operatorname{Ext}_{\mathbf{Z}}^*$ groups to get a calculational feel for what these derived functors do to abelian groups.

Lemma 3.3.1 $\operatorname{Ext}_{\mathbf{Z}}^n(A, B) = 0$ *for $n \geq 2$ and all abelian groups A, B.*

Proof. Embed B in an injective abelian group I^0; the quotient I^1 is divisible, hence injective. Therefore, $\operatorname{Ext}^*(A, B)$ is the cohomology of

$$0 \to \operatorname{Hom}(A, I^0) \to \operatorname{Hom}(A, I^1) \to 0.$$

$\square$

Calculation 3.3.2 ($A = {}^{\mathbf{Z}}\!/_{p}$) $\operatorname{Ext}_{\mathbf{Z}}^0\left({}^{\mathbf{Z}}\!/_{p}, B\right) = {}_pB$, $\operatorname{Ext}_{\mathbf{Z}}^1\left({}^{\mathbf{Z}}\!/_{p}, B\right) = {}^{B}\!/_{pB}$ and $\operatorname{Ext}_{\mathbf{Z}}^n\left({}^{\mathbf{Z}}\!/_{p}, B\right) = 0$ for $n \geq 2$. To see this, use the resolution

$$0 \to \mathbf{Z} \xrightarrow{p} \mathbf{Z} \to {}^{\mathbf{Z}}\!/_{p} \to 0 \text{ and the fact that } \operatorname{Hom}(\mathbf{Z}, B) \cong B$$

to see that $\operatorname{Ext}^*\left({}^{\mathbf{Z}}\!/_{p}, B\right)$ is the cohomology of $0 \leftarrow B \xleftarrow{p} B \leftarrow 0$.

Since $\mathbf{Z}$ is projective, $\operatorname{Ext}^1(\mathbf{Z}, B) = 0$. Hence we can calculate $\operatorname{Ext}^*(A, B)$ for every finitely generated abelian group $A \cong \mathbf{Z}^m \oplus {}^{\mathbf{Z}}\!/_{p_1} \oplus \cdots \oplus {}^{\mathbf{Z}}\!/_{p_n}$ by taking a finite direct sum of $\operatorname{Ext}^*\left({}^{\mathbf{Z}}\!/_{p}, B\right)$ groups. For infinitely generated groups, the calculation is much more complicated than it was for Tor.

Example 3.3.3 ($B = \mathbf{Z}$) Let A be a torsion group, and write A^* for its Pontrjagin dual $\operatorname{Hom}\left(A, \mathbf{Q}/\mathbf{Z}\right)$ as in 3.2.3. Using the injective resolution $0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0$ to compute $\operatorname{Ext}^*(A, \mathbf{Z})$, we see that $\operatorname{Ext}^0_{\mathbf{Z}}(A, \mathbf{Z}) = 0$ and $\operatorname{Ext}^1_{\mathbf{Z}}(A, \mathbf{Z}) = A^*$. To get a feel for this, note that because $\mathbf{Z}_{p^\infty}$ is the union (colimit) of its subgroups $\mathbf{Z}/p^n$, the group

$$\operatorname{Ext}^1_{\mathbf{Z}}(\mathbf{Z}_{p^\infty}, \mathbf{Z}) = (\mathbf{Z}_{p^\infty})^*$$

is the torsionfree group of p-adic integers, $\widehat{\mathbf{Z}}_p = \varprojlim \left(\mathbf{Z}/p^n\right)$. We will calculate $\operatorname{Ext}^1_{\mathbf{Z}}(\mathbf{Z}_{p^\infty}, B)$ more generally in section 3.5, using $\varprojlim^1$.

Exercise 3.3.1 Show that $\operatorname{Ext}^1_{\mathbf{Z}}\left(\mathbf{Z}\left[\frac{1}{p}\right], \mathbf{Z}\right) \cong \widehat{\mathbf{Z}}_p/\mathbf{Z} \cong \left(\mathbf{Q}/\mathbf{Z}\left[\frac{1}{p}\right]\right) \times \widehat{\mathbf{Q}}_p/\mathbf{Q}$. This shows that $\operatorname{Ext}^1(-, \mathbf{Z})$ does not vanish on flat abelian groups.

Recall from Example 2.3.3 that $\mathbf{Z}_{p^\infty} = \mathbf{Z}\left[\frac{1}{p}\right]/\mathbf{Z}$. Thus we have the short exact sequence

$$0 \to \mathbf{Z} \to \mathbf{Z}\left[\tfrac{1}{p}\right] \to \mathbf{Z}_{p^\infty} \to 0,$$

which gives rise to the long exact Ext sequence

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}(\mathbf{Z}_{p^\infty}, \mathbf{Z}) \arrow[r] & \operatorname{Hom}\left(\mathbf{Z}\left[\frac{1}{p}\right], \mathbf{Z}\right) \arrow[r] & \operatorname{Hom}(\mathbf{Z}, \mathbf{Z}) \arrow[dll, "\delta"] \\
 & \operatorname{Ext}^1_{\mathbf{Z}}(\mathbf{Z}_{p^\infty}, \mathbf{Z}) \arrow[r] & \operatorname{Ext}^1_{\mathbf{Z}}\left(\mathbf{Z}\left[\frac{1}{p}\right], \mathbf{Z}\right) \arrow[r] & \operatorname{Ext}^1_{\mathbf{Z}}(\mathbf{Z}, \mathbf{Z}) \arrow[dll, "\delta"] \\
 & \ddots & &
\end{tikzcd}$$

Now, by Example 3.3.3, $\operatorname{Ext}^1_{\mathbf{Z}}(\mathbf{Z}_{p^\infty}, \mathbf{Z}) \cong \widehat{\mathbf{Z}}_p$, and since $\mathbf{Z}$ is injective and Ext^* is a right derived functor, by 2.5.1, $\operatorname{Ext}^1_{\mathbf{Z}}(\mathbf{Z}, \mathbf{Z}) = 0$. Furthermore, $\operatorname{Hom}(\mathbf{Z}, \mathbf{Z}) \cong \mathbf{Z}$ as a basic algebra fact; a homomorphism $\mathbf{Z} \mapsto G$ is determined by the image of its generator 1, and mapping to codomain $\mathbf{Z}$ gives one such map for each image $z \in \mathbf{Z}$ of 1.

We claim that $\operatorname{Hom}\left(\mathbf{Z}\left[\frac{1}{p}\right], \mathbf{Z}\right) = 0$. With the claim assumed, we then have a short exact sequence

$$\begin{tikzcd}
 & 0 \arrow[r] & \mathbf{Z} \arrow[dll, "\delta"] \\
\widehat{\mathbf{Z}}_p \arrow[r] & \operatorname{Ext}^1_{\mathbf{Z}}\left(\mathbf{Z}\left[\frac{1}{p}\right], \mathbf{Z}\right) \arrow[r] & 0,
\end{tikzcd}$$

and therefore $\operatorname{Ext}^1_{\mathbf{Z}}\left(\mathbf{Z}\left[\frac{1}{p}\right], \mathbf{Z}\right) \cong \widehat{\mathbf{Z}}_p/\mathbf{Z}$, as we need to show. So, to prove the claim, suppose $f \in \operatorname{Hom}\left(\mathbf{Z}\left[\frac{1}{p}\right], \mathbf{Z}\right)$ is any map, and that $f(1) = z \in \mathbf{Z}$ for an arbitrary z. As f is a homomorphism, for all $n \in \mathbf{N}$,

$$f\left(\frac{1}{p^n}\right) = \frac{1}{p^n} f(1) = \frac{1}{p^n} z \in \mathbf{Z}$$

As a nonzero integer can only have finitely many factors p in its prime factor decomposition, $\frac{z}{p^n} = 0$, so $z = 0$, and thus f is the zero map. Thus the claim is shown.

Now, we show that $\widehat{\mathbf{Z}}_p/\mathbf{Z} \cong \mathbf{Q}/\mathbf{Z}\left[\frac{1}{p}\right] \times \widehat{\mathbf{Q}}_p/\mathbf{Q}$.

Exercise 3.3.2 When $R = \mathbf{Z}/m$ and $B = \mathbf{Z}/p$ with $p \mid m$, show that

$$0 \to \mathbf{Z}/p \xhookrightarrow{\iota} \mathbf{Z}/m \xrightarrow{p} \mathbf{Z}/m \xrightarrow{\frac{m}{p}} \mathbf{Z}/m \xrightarrow{p} \mathbf{Z}/m \xrightarrow{\frac{m}{p}} \cdots$$

is an infinite periodic injective resolution of B. Then compute the groups $\operatorname{Ext}^n_{\mathbf{Z}/m}\left(A, \mathbf{Z}/p\right)$ in terms of $A^* = \operatorname{Hom}\left(A, \mathbf{Z}/m\right)$. In particular, show that if $p^2 \mid m$, then $\operatorname{Ext}^n_{\mathbf{Z}/m}\left(\mathbf{Z}/p, \mathbf{Z}/p\right) \cong \mathbf{Z}/p$ for all n.

The sequence is infinite and periodic, injective as $\mathbf{Z}/m\mathbf{Z}$ is injective by Exercise 2.3.1, so it only remains to show that the sequence is a resolution, i.e., exact. Observe that

$$\begin{aligned}
\ker(p) &= \{[x]_m \mid p[x]_m = [px]_m = 0\} = \tfrac{m}{p}\mathbf{Z}/m\mathbf{Z}, \\
\operatorname{im}\left(\tfrac{m}{p}\right) &= \tfrac{m}{p}\mathbf{Z}/m\mathbf{Z}, \\
\ker\left(\tfrac{m}{p}\right) &= \left\{[x]_m \mid \tfrac{m}{p}[x]_m = \left[\tfrac{mx}{p}\right]_m = 0\right\} = p\mathbf{Z}/m\mathbf{Z}, \text{ and} \\
\operatorname{im}(p) &= p\mathbf{Z}/m\mathbf{Z},
\end{aligned}$$

so the sequence is exact.

Now we use this injective resolution to compute $\operatorname{Ext}^n_{\mathbf{Z}/m\mathbf{Z}}\left(A, \mathbf{Z}/p\mathbf{Z}\right)$ in terms of $A^* = \operatorname{Hom}\left(A, \mathbf{Z}/m\mathbf{Z}\right)$. We must compute the cohomology of

$$0 \to \operatorname{Hom}\left(A, \mathbf{Z}/m\mathbf{Z}\right) \xrightarrow{p^*} \operatorname{Hom}\left(A, \mathbf{Z}/m\mathbf{Z}\right) \xrightarrow{\left(\frac{m}{p}\right)^*} \operatorname{Hom}\left(A, \mathbf{Z}/m\mathbf{Z}\right) \xrightarrow{p^*} \operatorname{Hom}\left(A, \mathbf{Z}/m\mathbf{Z}\right) \xrightarrow{\left(\frac{m}{p}\right)^*} \cdots$$

i.e.,

$$0 \longrightarrow A^* \xrightarrow{\quad p^* \quad} A^* \xrightarrow{\quad \left(\frac{m}{p}\right)^* \quad} A^* \xrightarrow{\quad p^* \quad} A^* \xrightarrow{\left(\frac{m}{p}\right)^*} \cdots$$

Hence by definition,

$$\mathrm{Ext}^n_{\mathbf{Z}/m\mathbf{Z}}\left(A, \mathbf{Z}/m\mathbf{Z}\right) = \begin{cases} A^* = \mathrm{Hom}\left(A, \mathbf{Z}/m\mathbf{Z}\right) & \text{if } n = 0, \\ \ker\left(\frac{m}{p}\right)^* \big/ \mathrm{im}\, p^* & \text{if } n = 2k+1, k \in \mathbf{N}, \\ \ker p^* \big/ \mathrm{im}\left(\frac{m}{p}\right)^* & \text{if } n = 2k, k \in \mathbf{N}, \\ 0 & \text{otherwise,} \end{cases}$$

where $p^*\left(A \xrightarrow{f} \mathbf{Z}/m\mathbf{Z}\right) = p \circ f$, and similarly $\left(\frac{m}{p}\right)^*(f) = \frac{m}{p} \circ f$. Once A is determined, these groups may be computed.

In the case that $A = \mathbf{Z}/p\mathbf{Z}$ and p^2 divides m, then see that we may compute $\mathrm{Ext}^*_{\mathbf{Z}/m\mathbf{Z}}\left(A, \mathbf{Z}/m\mathbf{Z}\right)$ as the cohomology of

$$0 \longrightarrow \mathrm{Hom}\left(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/m\mathbf{Z}\right) \xrightarrow{p^*} \mathrm{Hom}\left(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/m\mathbf{Z}\right) \xrightarrow{\left(\frac{m}{p}\right)^*} \mathrm{Hom}\left(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/m\mathbf{Z}\right) \xrightarrow{p^*} \cdots$$

We first need to compute $\mathrm{Hom}\left(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/m\mathbf{Z}\right)$. Since p^2 divides m, $m = p^k n$ for $k \geq 2$ and n coprime to p. Hence we may write

$$\begin{aligned} \mathrm{Hom}\left(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/m\mathbf{Z}\right) = \mathrm{Hom}\left(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p^k n\mathbf{Z}\right) &\cong \mathrm{Hom}\left(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p^k\mathbf{Z} \oplus \mathbf{Z}/n\mathbf{Z}\right) \\ &\cong \mathrm{Hom}\left(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p^k\mathbf{Z}\right) \oplus \mathrm{Hom}\left(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/n\mathbf{Z}\right), \end{aligned}$$

as we may commute products out of the second factor of Hom, and finite products and finite direct sums agree. We now claim $\mathrm{Hom}\left(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/n\mathbf{Z}\right) = 0$. To see this is a routine algebra exercise; let f be a map $f : \mathbf{Z}/p\mathbf{Z} \to \mathbf{Z}/n\mathbf{Z}$ defined by $f([1]_p) = [t]_n$. It must be the case that

$$0 = f([0]_p) = f([p]_p) = pf([1]_p) = p[t]_n = [pt]_n,$$

so pt is a multiple of n. Yet, by assumption, $\gcd(p, n) = 1$, so t must be zero, and thus $\mathrm{Hom}\left(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/n\mathbf{Z}\right) = 0$, as claimed. So we are reduced to computing $\mathrm{Hom}\left(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p^k\mathbf{Z}\right)$. We claim that $\mathrm{Hom}\left(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p^k\mathbf{Z}\right) \cong \mathbf{Z}/p\mathbf{Z}$. To see this, again, realize that a map $f : \mathbf{Z}/p\mathbf{Z} \to \mathbf{Z}/p^k\mathbf{Z}$ is determined by the image of $[1]_p$; call it $[t]_{p^k}$. Again, see that

$$0 = f([0]_p) = f([p]_p) = pf([1]_p) = p[t]_{p^k} = [pt]_{p^k},$$

so pt must be a multiple of p^k; i.e., t must be a multiple of p^{k-1}. There are p such elements in $\mathbf{Z}/p^k\mathbf{Z}$. Thus, $\operatorname{Hom}\left(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/m\mathbf{Z}\right) \cong \mathbf{Z}/p\mathbf{Z}$. Our complex is now

$$0 \longrightarrow \mathbf{Z}/p\mathbf{Z} \xrightarrow{\quad p \quad} \mathbf{Z}/p\mathbf{Z} \xrightarrow{\frac{m}{p}=p^{k-1}n} \mathbf{Z}/p\mathbf{Z} \xrightarrow{\quad p \quad} \mathbf{Z}/p\mathbf{Z} \xrightarrow{p^{k-1}n} \cdots$$

and we know that multiplying by at least p will send every element to 0 in $\mathbf{Z}/p\mathbf{Z}$, so since $k \geq 2$, that means $k-1 \geq 1$, so every map is indeed multiplication by at least p, and therefore

$$\operatorname{Ext}^n_{\mathbf{Z}/m\mathbf{Z}}\left(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/m\mathbf{Z}\right) = \begin{cases} \left(\mathbf{Z}/p\mathbf{Z}\right)^* = \operatorname{Hom}\left(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/m\mathbf{Z}\right) \cong \mathbf{Z}/p\mathbf{Z} & \text{if } n = 0, \\ \ker(p^{k-1}n)/\operatorname{im} p = \left(\mathbf{Z}/p\mathbf{Z}\right)/0 \cong \mathbf{Z}/p\mathbf{Z} & \text{if } n \text{ is odd}, \\ \ker p/\operatorname{im}(p^{k-1}n) = \left(\mathbf{Z}/p\mathbf{Z}\right)/0 \cong \mathbf{Z}/p\mathbf{Z} & \text{if } n \text{ is even}, \\ 0 & \text{otherwise}, \end{cases}$$

as we wished to show.

Proposition 3.3.4 *For all n and all rings R*

1. $\operatorname{Ext}^n_R(\oplus_\alpha A_\alpha, B) \cong \prod_\alpha \operatorname{Ext}^n_R(A_\alpha, B)$.
2. $\operatorname{Ext}^n_R(A, \prod_\beta B_\beta) \cong \prod_\beta \operatorname{Ext}^n_R(A, B_\beta)$.

Proof. If $P_\alpha \to A_\alpha$ are projective resolutions, so is $\oplus P_\alpha \to \oplus A_\alpha$. If $B_\beta \to I_\beta$ are injective resolutions, so is $\prod B_\beta \to \prod I_\beta$. Since $\operatorname{Hom}(\oplus P_\alpha, B) = \prod \operatorname{Hom}(P_\alpha, B)$ and $\operatorname{Hom}(A, \prod I_\beta) = \prod \operatorname{Hom}(A, I_\beta)$, the result follows from the fact that for any family C_γ of cochain complexes,

$$H^*(\prod C_\gamma) \cong \prod H^*(C_\gamma).$$

$\square$

Examples 3.3.5

1. If $p^2 \mid m$ and A is a $\mathbf{Z}/p$-vector space of countably infinite dimension, then $\operatorname{Ext}^n_{\mathbf{Z}/m}\left(A, \mathbf{Z}/p\right) \cong \prod_{i=1}^\infty \mathbf{Z}/p$ is a $\mathbf{Z}/p$-vector space of dimension $2^{\aleph_0}$.
2. If B is the product $\mathbf{Z}/2 \times \mathbf{Z}/3 \times \mathbf{Z}/4 \times \mathbf{Z}/5 \times \cdots$ then B is *not* a torsion group, and
$$\operatorname{Ext}^1(A, B) = \prod_{p=2}^{\infty} A^*/pA^*$$
vanishes if and only if A^* is divisible, i.e., A is torsionfree.

Lemma 3.3.6 *Suppose that R is a commutative ring, so that* $\operatorname{Hom}_R(A, B)$ *and the* $\operatorname{Ext}^*_R(A, B)$ *are actually R-modules. If $\mu : A \to A$ and $\nu : B \to B$ are multiplication by $r \in R$, so are the induced endomorphisms μ^* and ν_* of* $\operatorname{Ext}^n_R(A, B)$ *for all n.*

Proof. Pick a projective resolution $P \to A$. Multiplication by r is an R-module chain map $\widetilde{\mu} : P \to P$ over μ (as r is central); the map $\operatorname{Hom}(\widetilde{\mu}, B)$ on $\operatorname{Hom}(P, B)$ is multiplication by r, because it sends $f \in \operatorname{Hom}(P_n, B)$ to $f\widetilde{\mu}$, which takes $p \in P_n$ to $f(rp) = rf(p)$. Hence the map μ^* on the subquotient $\operatorname{Ext}^n(A, B)$ of $\operatorname{Hom}(P_n, B)$ is also multiplication by r. The argument for ν_* is similar, using an injective resolution $B \to I$. $\square$

Corollary 3.3.7 *If R is commutative and A is actually an R/r-module, then for every R-module B the R-modules $\operatorname{Ext}^*_R(A,B)$ are actually R/r-modules.*

We would like to conclude, as we did for Tor, that Ext commutes with localization in some sense. Indeed, there is a natural map Φ from $S^{-1}\operatorname{Hom}_R(A,B)$ to $\operatorname{Hom}_{S^{-1}R}(S^{-1}A, S^{-1}B)$, but it need not be ana isomorphism. A sufficient condition is that A be finitely presented, that is, some $R^m \xrightarrow{\alpha} R^n \to A \to 0$ is exact.

Lemma 3.3.8 *If A is a finitely presented R-module, then for every central multiplicative set S in R, Φ is an isomorphism:*

$$\Phi : S^{-1}\operatorname{Hom}_R(A,B) \cong \operatorname{Hom}_{S^{-1}R}(S^{-1}A, S^{-1}B).$$

Proof. Φ is trivially an isomorphism when $A = R$; as Hom is additive, Φ is also an isomorphism when $A = R^m$. The result now follows from the 5-lemma and the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & S^{-1}\operatorname{Hom}_R(A,B) \arrow[r] \arrow[d, "\Phi"'] & S^{-1}\operatorname{Hom}_R(R^n,B) \arrow[r, "\alpha"] \arrow[d, "\cong"'] & S^{-1}\operatorname{Hom}_R(R^m,B) \arrow[d, "\cong"'] \\
0 \arrow[r] & \operatorname{Hom}(S^{-1}A, S^{-1}B) \arrow[r] & \operatorname{Hom}(S^{-1}R^n, S^{-1}B) \arrow[r, "\alpha"] & \operatorname{Hom}(S^{-1}R^m, S^{-1}B).
\end{tikzcd}$$

□

Definition 3.3.9 A ring R is *(right) noetherian* if every (right) ideal is finitely generated, that is, if every module R/I is finitely presented. It is well known that if R is noetherian, then every finitely generated (right) R-module is finitely presented. (See [BAII§3.2].) It follows that every finitely generated module A has a resolution $F \to A$ in which each F_n is a finitely generated free R-module.

Proposition 3.3.10 *Let A be a finitely generated module over a commutative noetherian ring R. Then for every multiplicative set S, all modules B, and all n*

$$\Phi : S^{-1}\operatorname{Ext}^n_R(A,B) \cong \operatorname{Ext}^n_{S^{-1}R}(S^{-1}A, S^{-1}B).$$

Proof. Choose a resolution $F \to A$ by finitely generated free R-modules. Then $S^{-1}F \to S^{-1}A$ is a resolution by finitely generated free $S^{-1}R$-modules. Because S^{-1} is an exact functor from R-modules to $S^{-1}R$-modules,

$$\begin{aligned} S^{-1}\operatorname{Ext}^*_R(A,B) &= S^{-1}(H^*\operatorname{Hom}_R(F,B)) \cong H^*(S^{-1}\operatorname{Hom}_R(F,B)) \\ &\cong H^*\operatorname{Hom}_{S^{-1}R}(S^{-1}F, S^{-1}B) = \operatorname{Ext}^*_{S^{-1}R}(S^{-1}A, S^{-1}B). \end{aligned}$$

□

Corollary 3.3.11 (Localization for Ext) *If R is commutative noetherian and A is a finitely generated R-module, then the following are equivalent for all modules B and all n:*

1. $\operatorname{Ext}^n_R(A,B) = 0$.
2. *For every prime ideal p of R,* $\operatorname{Ext}^n_{R_p}(A_p, B_p) = 0$.
3. *For every maximal ideal m of R,* $\operatorname{Ext}^n_{R_m}(A_m, B_m) = 0$.

3.4 Ext and Extensions

An *extension* ξ of A by B is an exact sequence $0 \to B \to X \to At \to 0$. Two extensions ξ and ξ' are *equivalent* if there is a commutative diagram

$$\begin{tikzcd}
\xi : 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\cong"'] & A \arrow[r] \arrow[d, equal] & 0 \\
\xi' : 0 \arrow[r] & B \arrow[r] & X' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

An extension is *split* if it is equivalent to $0 \to B \xrightarrow{(0,1)} A \oplus B \to A \to 0$.

Exercise 3.4.1 Show that if p is prime, there are exactly p equivalence classes of extensions of $\mathbf{Z}/p$ by $\mathbf{Z}/p$ in **Ab**: the split extension and the extensions

$$0 \to \mathbf{Z}/p \xrightarrow{p} \mathbf{Z}/p^2 \xrightarrow{i} \mathbf{Z}/p \to 0 \qquad\qquad (i = 1, 2, \cdots, p-1).$$

Let E be an arbitrary extension of $\mathbf{Z}/p\mathbf{Z}$ by $\mathbf{Z}/p\mathbf{Z}$ in **Ab**; i.e.,

$$0 \to \mathbf{Z}/p\mathbf{Z} \to E \to \mathbf{Z}/p\mathbf{Z} \to 0$$

is a short exact sequence. Hence $|E| = \left|\mathbf{Z}/p\mathbf{Z}\right| \cdot \left|\mathbf{Z}/p\mathbf{Z}\right| = p^2$, and by the classification of finite abelian groups, this means that either $E \cong \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}$, or $E \cong \mathbf{Z}/p^2\mathbf{Z}$.

In the case that $E \cong \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}$, we claim that the only such extension is the split extension. To see this, let $\mathbf{Z}/p\mathbf{Z} \to \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}$ be defined by mapping $1 \mapsto (a, b)$. Similarly, by the surjectivity of $\mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z} \to \mathbf{Z}/p\mathbf{Z}$, choose a preimage (c, d) of 1 so that $(c, d) \mapsto 1$. Observe that $\langle (a, b), (c, d) \rangle \cong \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}$, because $(c, d) \mapsto 1$ and hence $(c, d) \notin \ker\left(\mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z} \to \mathbf{Z}/p\mathbf{Z}\right) = \langle (a, b)\rangle$, and as a field $\dim\left(\mathbf{Z}/p\mathbf{Z}^{2}\right) = 2$, so two linearly independent elements form its basis.

Now define $\sigma : \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z} \to \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}$ by mapping $\sigma(1, 0) = (a, b)$ and $\sigma(0, 1) = (c, d)$. Since also $\langle (1, 0), (0, 1)\rangle \cong \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}$, σ is an isomorphism. Thus, the following diagram commutes,

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z}/p\mathbf{Z} \arrow[r, "1 \mapsto (1,0)"'] \arrow[d, equal] & \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z} \arrow[r, "(0,1) \mapsto 1"'] \arrow[d] & \mathbf{Z}/p\mathbf{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbf{Z}/p\mathbf{Z} \arrow[r, "1 \mapsto (a,b)"] & \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z} \arrow[r, "(c,d) \mapsto 1"] & \mathbf{Z}/p\mathbf{Z} \arrow[r] & 0
\end{tikzcd}$$

so by definition, our arbitrarily constructed extension is equivalent to the split extension.

Now consider the case that $E \cong \mathbf{Z}/p^2\mathbf{Z}$ and we have an arbitrary extension

$$0 \to \mathbf{Z}/p\mathbf{Z} \xrightarrow{f} \mathbf{Z}/p^2\mathbf{Z} \xrightarrow{g} \mathbf{Z}/p\mathbf{Z} \to 0.$$

Since $\mathbf{Z}/p\mathbf{Z} \to \mathbf{Z}/p^2\mathbf{Z}$ is an injection, its image must be the only subgroup of $\mathbf{Z}/p^2\mathbf{Z}$ of order p, which is $p\mathbf{Z}/p^2\mathbf{Z}$. Thus define the map via $1 \mapsto pa$, and note that p cannot divide a. Further, define the map $\mathbf{Z}/p^2\mathbf{Z} \to \mathbf{Z}/p\mathbf{Z}$ by $1 \mapsto b$. We build a commutative diagram; consider the map $\sigma : \mathbf{Z}/p^2\mathbf{Z} \to \mathbf{Z}/p^2\mathbf{Z}$ defined by $\sigma(1) = a^{-1}$. As p does not divide a, σ is a well-defined isomorphism. Our diagram is therefore

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{Z}/p\mathbf{Z} \arrow[r, "f"] \arrow[d, equal] & \mathbf{Z}/p^2\mathbf{Z} \arrow[r, "g"] \arrow[d, "\sigma"] & \mathbf{Z}/p\mathbf{Z} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbf{Z}/p\mathbf{Z} \arrow[r, "p"] & \mathbf{Z}/p^2\mathbf{Z} \arrow[r, "ab"] & \mathbf{Z}/p\mathbf{Z} \arrow[r] & 0 \end{tikzcd}$$

and if we write $i = ab$, we see that our extension is one of the $0 \to \mathbf{Z}/p\mathbf{Z} \xrightarrow{p} \mathbf{Z}/p^2\mathbf{Z} \xrightarrow{i} \mathbf{Z}/p\mathbf{Z} \to 0$, $i \in \{1, ..., p-1\}$, provided. It only remains to show that if $i \neq j$, then $0 \to \mathbf{Z}/p\mathbf{Z} \xrightarrow{p} \mathbf{Z}/p^2\mathbf{Z} \xrightarrow{i} \mathbf{Z}/p\mathbf{Z} \to 0$ is not equivalent to $0 \to \mathbf{Z}/p\mathbf{Z} \xrightarrow{p} \mathbf{Z}/p^2\mathbf{Z} \xrightarrow{j} \mathbf{Z}/p\mathbf{Z} \to 0$, so that there are exactly p extensions.

To see this, we show the contrapositive. Suppose we do have an equivalence of extensions given by the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{Z}/p\mathbf{Z} \arrow[r, "p"] \arrow[d, equal] & \mathbf{Z}/p^2\mathbf{Z} \arrow[r, "i"] \arrow[d, "\sigma"] & \mathbf{Z}/p\mathbf{Z} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbf{Z}/p\mathbf{Z} \arrow[r, "p"] & \mathbf{Z}/p^2\mathbf{Z} \arrow[r, "j"] & \mathbf{Z}/p\mathbf{Z} \arrow[r] & 0 \end{tikzcd}$$

Since the right square is commutative, $\sigma(1) \equiv ij^{-1} \pmod{p}$. Since the left square is commutative, $\sigma(pa) = pa$ for all $a \in \mathbf{Z}/p\mathbf{Z}$. Therefore,

$$pa = \sigma(pa) = \sigma(1 \cdot pa) = \sigma(1)\sigma(pa) \equiv ij^{-1}pa \pmod{p},$$

so $1 \equiv ij^{-1} \pmod{p}$, and thus $i \equiv j \pmod{p}$, as we wished to show.

Lemma 3.4.1 *If* $\mathrm{Ext}^1(A, B) = 0$*, then every extension of* A *by* B *is split.*

Proof. Given an extension ξ, applying $\mathrm{Ext}^*(-, B)$ yields the exact sequence

$$\mathrm{Hom}(X, B) \to \mathrm{Hom}(B, B) \xrightarrow{\partial} \mathrm{Ext}^1(A, B)$$

so the identity map id_B lifts to a map $\sigma : X \to B$ when $\mathrm{Ext}^1(A, B) = 0$. As σ is a section of $B \to X$, evidently $X \cong A \oplus B$ and ξ is split. □

Porism 3.4.2 Taking the construction of this lemma to heart, we see that the class $\Theta(\xi) = \partial(\mathrm{id}_B)$ in $\mathrm{Ext}^1(A, B)$ is an *obstruction* to ξ being split: ξ is split iff id_B lifts to $\mathrm{Hom}(X, B)$ iff the class $\Theta(\xi) \in \mathrm{Ext}^1(A, B)$ vanishes. Equivalent extensions have the same obstruction by naturality of the map ∂, so the obstruction $\Theta(\xi)$ only depends on the equivalence class of ξ.

Theorem 3.4.3 *Given two R-modules A and B, the mapping $\Theta : \xi \mapsto \partial(\mathrm{id}_B)$ establishes a 1-1 correspondence*

$$\left\{ \begin{array}{c} \text{equivalence classes of} \\ \text{extensions of } A \text{ by } B \end{array} \right\} \overset{1-1}{\longleftrightarrow} \mathrm{Ext}^1(A, B)$$

in which the split extension corresponds to the element $0 \in \mathrm{Ext}^1(A, B)$.

Proof. Fix an exact sequence $0 \to M \xrightarrow{j} P \to A \to 0$ with P projective. Applying $\mathrm{Hom}(-, B)$ yields an exact sequence

$$\mathrm{Hom}(P, B) \to \mathrm{Hom}(M, B) \xrightarrow{\partial} \mathrm{Ext}^1(A, B) \to 0.$$

Given $x \in \mathrm{Ext}^1(A, B)$, choose $\beta \in \mathrm{Hom}(M, B)$ with $\partial(\beta) = x$. Let X be the pushout of j and β, i.e., the cokernel of $M \to P \oplus B$ $(m \mapsto (j(m), -\beta(m)))$. There is a diagram

$$\begin{tikzcd}
& 0 \arrow[r] & M \arrow[r, "j"] \arrow[d, "\beta"] \arrow[dr, phantom, "\lrcorner", very near end] & P \arrow[r] \arrow[d, "\sigma"] & A \arrow[r] \arrow[d, equal] & 0 \\
\xi : & 0 \arrow[r] & B \arrow[r, "i"] & X \arrow[r] & A \arrow[r] & 0,
\end{tikzcd}$$

where the map $X \to A$ is induced by the maps $B \xrightarrow{0} A$ and $P \to A$. (*Exercise*: Show that the bottom sequence ξ is exact.) By naturality of the connecting map ∂, we see that $\Theta(\xi) = x$, that is, that Θ is surjection.

In fact, this construction gives a set map Ψ from $\mathrm{Ext}^1(A, B)$ to the set of equivalence classes of extensions. For if $\beta' \in \mathrm{Hom}(M, B)$ is another lift of x, then there is an $f \in \mathrm{Hom}(P, B)$ so that $\beta' = \beta + fj$. If X' is the pushout of j and β', then the maps $i : B \to X$ and $\sigma + if : P \to X$ induce an isomorphism $X' \cong X$ and an equivalence between ξ' and ξ. (Check this!)

Conversely, given an extension ξ of A by B, the lifting property of P gives a map $\tau : P \to X$ and hence a commutative diagram

$$\begin{tikzcd}
& 0 \arrow[r] & M \arrow[r, "j"] \arrow[d, "\gamma"] & P \arrow[r] \arrow[d, "\tau"] & A \arrow[r] \arrow[d, equal] & 0 \\
\xi : & 0 \arrow[r] & B \arrow[r, "i"] & X \arrow[r] & A \arrow[r] & 0.
\end{tikzcd} \tag{*}$$

Now X is the pushout of j and γ. (*Exercise*: Check this!) Hence $\Psi(\Theta(\xi)) = \xi$, showing that Θ is injective. $\square$

Definition 3.4.4 (Baer sum) Let $\xi : 0 \to B \to X \to A \to 0$ and $\xi' : 0 \to B \to X' \to A \to 0$ be two extensions of A by B. Let X'' be the pullback $\{(x, x') \in X \times X' \mid \overline{x} = \overline{x'} \text{ in } A\}$.

$$\begin{tikzcd}
X'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X' \arrow[d] \\
X \arrow[r] & A
\end{tikzcd}$$

X'' contains three copies of B: $B \times 0$, $0 \times B$, and the skew diagonal $\{(-b, b) \mid b \in B\}$. The copies $B \times 0$ and $0 \times B$ are identified in the quotient Y of X'' by the skew diagonal. Since $X''/_{0 \times B} \cong X$ and $X/_B \cong A$, it is immediate that the sequence

$$\varphi : 0 \to B \to Y \to A \to 0$$

is also an extension of A by B. The class of φ is called the *Baer sum* of the extensions ξ and ξ', since this construction was introduced by R. Baer in 1934.

Corollary 3.4.5 *The set of (equiv. classes of) extensions is an abelian group under Baer sum, with zero being the class of the split extension. The map Θ is an isomorphism of abelian groups.*

Proof. We will show that $\Theta(\varphi) = \Theta(\xi) + \Theta(\xi')$ in $\mathrm{Ext}^1(A,B)$. This will prove that Baer sum is well defined up to equivalence, and the corollary will then follow. We shall adopt the notation used in (*) in the proof of the above theorem. Let $\tau'' : P \to X''$ be the map induced by $\tau : P \to X$ and $\tau' : P \to X'$, and let $\overline{\tau} : P \to Y$ be the induced map. The restriction of $\overline{\tau}$ to M is induced by the map $\gamma + \gamma' : M \to B$, so

$$\begin{array}{ccccccccc}
 & 0 & \longrightarrow & M & \longrightarrow & P & \longrightarrow & A & \longrightarrow & 0 \\
 & & & \Big\downarrow{\scriptstyle \gamma+\gamma'} & & \Big\downarrow{\scriptstyle \overline{\tau}} & & \Big\| & & \\
\varphi : & 0 & \longrightarrow & B & \longrightarrow & Y & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

commutes. Hence, $\Theta(\varphi) = \partial(\gamma + \gamma')$, where ∂ is the map from $\mathrm{Hom}(M,B)$ to $\mathrm{Ext}^1(A,B)$. But $\partial(\gamma+\gamma') = \partial(\gamma) + \partial(\gamma') = \Theta(\xi) + \Theta(\xi')$. ☐

Vista 3.4.6 (Yoneda Ext groups) We can define $\mathrm{Ext}^1(A,B)$ in *any* abelian category $\mathcal{A}$, even if it has no projectives and no injectives, to be the set of equivalence classes of extensions under Baer sum (if indeed this is a set). The Freyd-Mitchell Embedding Theorem 1.6.1 shows that $\mathrm{Ext}^1(A,B)$ is an abelian group-but one could also prove this fact directly. Similarly, we can recapture the groups $\mathrm{Ext}^n(A,B)$ without mentioning projectives or injectives. This approach is due to Yoneda. An element of the Yoneda $\mathrm{Ext}^n(A,B)$ is an equivalence class of exact sequences of the form

$$\xi : 0 \to B \to X_n \to \cdots \to X_1 \to A \to 0.$$

The equivalence relation is generated by the relation that $\xi' \sim \xi''$ if there is a diagram

$$\begin{array}{cccccccccccc}
\xi' : & 0 & \longrightarrow & B & \longrightarrow & X_n{}' & \longrightarrow & \cdots & \longrightarrow & X_1{}' & \longrightarrow & A & \longrightarrow & 0 \\
 & & & \Big\| & & \Big\downarrow & & & & \Big\downarrow & & \Big\| & & \\
\xi'' : & 0 & \longrightarrow & B & \longrightarrow & X_n{}'' & \longrightarrow & \cdots & \longrightarrow & X_1{}'' & \longrightarrow & A & \longrightarrow & 0.
\end{array}$$

To "add" ξ and ξ' when $n \geq 2$, let $X_1{}''$ be the pullback of X_1 and $X_1{}'$ over A, let $X_n{}''$ be the pushout of X_n and $X_n{}'$ under B, and let Y_n be the quotient of $X_n{}''$ by the skew diagonal copy of B. Then $\xi + \xi'$ is the class of the extension

$$0 \to B \to X_n{}'' \to X_{n-1} \oplus X_{n-1}{}' \to \cdots \to X_2 \oplus X_2{}' \to X_1{}'' \to A \to 0.$$

Now suppose that $\mathcal{A}$ has enough projectives. If $P \to A$ is a projective resolution, the Comparison Theorem 2.2.6 yields a map from P to ξ, hence a diagram

$$\begin{array}{cccccccccccc}
 & 0 & \longrightarrow & M & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\
 & & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \gamma_n} & & & & \Big\downarrow & & \Big\| & & \\
\xi : & 0 & \longrightarrow & B & \longrightarrow & X_n & \longrightarrow & \cdots & \longrightarrow & X_1 & \longrightarrow & A & \longrightarrow & 0.
\end{array}$$

By dimension shifting, there is an exact sequence

$$\mathrm{Hom}(P_{n-1}, B) \to \mathrm{Hom}(M,B) \xrightarrow{\partial} \mathrm{Ext}^n(A,B) \to 0.$$

The association $\Theta(\xi) = \partial(\beta)$ gives the 1-1 correspondence between the Yoneda Ext^n and the derived functor Ext^n. For more details we refer the reader to [BX, §7.5] or [MacH, pp. 82-87].

3.5 Derived Functors of the Inverse Limit

Let I be a small category and $\mathcal{A}$ an abelian category. We saw in Chapter 2 that the functor category $\mathcal{A}^I$ has enough injectives, at least when $\mathcal{A}$ is complete and has enough injectives. (For example, $\mathcal{A}$ could be **Ab**, R-**mod**, or Sheaves(X).) Therefore we can define the right derived functors $R^n \lim_{i\in I}$ from $\mathcal{A}^I$ to $\mathcal{A}$.

We are most interested in the case in which $\mathcal{A}$ is **Ab** and I is the poset $\cdots 2 \to 1 \to 0$ of whole numbers in reverse order. We shall call the objects of $\mathbf{Ab}^I$ (countable) *towers* of abelian groups; they have the form

$$\{A_i\}: \quad \cdots \to A_2 \to A_1 \to A_0.$$

In this section we shall give the alternative construction $\varprojlim^1$ of $R^1 \varprojlim$ for countable towers due to Eilenberg and prove that $R^n \varprojlim = 0$ for $n \neq 0, 1$. This construction generalizes from **Ab** to other abelian categories that satisfy the following axiom, introduced by Grothendieck in [Tohoku]:

$(AB4^*)$ $\mathcal{A}$ is complete, and the product of any set of surjections is a surjection.

Explanation If I is a discrete set, $\mathcal{A}^I$ is the product category $\prod_{i\in I} \mathcal{A}$ of indexed families of objects $\{A_i\}$ in $\mathcal{A}$. For $\{A_i\}$ in $\mathcal{A}^I$, $\lim_{i\in I} A_i$ is the product $\prod A_i$. Axiom $(AB4^*)$ states that the left exact functor $\prod$ from $\mathcal{A}^I$ to $\mathcal{A}$ is exact for all discrete I. Axiom $(AB4^*)$ fails ($\prod_{i=1}^{\infty}$ is not exact) for some important abelian categories, such as Sheaves(X). On the other hand, axiom $(AB4^*)$ is satisfied by many abelian categories in which objects have underlying sets, such as **Ab**, **mod**-R, and **Ch**(**mod**-R).

Definition 3.5.1 Given a tower $\{A_i\}$ in **Ab**, define the map

$$\Delta : \prod_{i=0}^{\infty} A_i \to \prod_{i=0}^{\infty} A_i$$

by the element-theoretic formula

$$\Delta(\cdots, .a_i, \cdots, a_0) = (\cdots, a_i - \overline{a_{i+1}}, \cdots, a_1 - \overline{a_2}, a_0 - \overline{a_1}),$$

where $\overline{a_{i+1}}$ denotes the image of $a_{i+1} \in A_{i+1}$ in A_i. The kernel of Δ is $\varprojlim A_i$ (check this!). We define $\varprojlim^1 A_i$ to be the cokernel of Δ, so that $\varprojlim^1$ is a functor from $\mathbf{Ab}^I$ to **Ab**. We also set $\varprojlim^0 A_i = \varprojlim A_i$ and $\varprojlim^n A_i = 0$ for $n \neq 0, 1$.

Lemma 3.5.2 *The functors $\{\varprojlim^n\}$ form a cohomological δ-functor.*

Proof. If $0 \to \{A_i\} \to \{B_i\} \to \{C_i\} \to 0$ is a short exact sequence of towers, apply the Snake Lemma to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \prod A_i & \longrightarrow & \prod B_i & \longrightarrow & \prod C_i & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta} & & \\ 0 & \longrightarrow & \prod A_i & \longrightarrow & \prod B_i & \longrightarrow & \prod C_i & \longrightarrow & 0 \end{array}$$

to get the requisite natural long exact sequence. □

Lemma 3.5.3 *If all the maps $A_{i+1} \to A_i$ are onto, then $\varprojlim^1 A_i = 0$. Moreover $\varprojlim A_i \neq 0$ (unless every $A_i = 0$), because each of the natural projections $\varprojlim A_i \to A_j$ are onto.*

Proof. Given elements $b_i \in A_i$ $(i = 0, 1, \cdots)$, and any $a_0 \in A_0$, inductively choose $a_{i+1} \in A_{i+1}$ to be a lift of $a_i - b_i \in A_i$. The map Δ sends $(\cdots, a_1, a_0)$ to $(\cdots, b_1, b_0)$, so Δ is onto and coker$(\Delta) = 0$. If all the $b_i = 0$, then $(\cdots, a_1, a_0) \in \varprojlim A_i$. □

Corollary 3.5.4 $\varprojlim^1 A_i \cong (R^1 \varprojlim)(A_i)$ *and* $R^n \varprojlim = 0$ *for* $n \neq 0, 1$.

Proof. In order to show that the $\varprojlim^n$ forms a universal δ-functor, we only need to see that $\varprojlim^1$ vanishes on enough injectives. In Chapter 2 we constructed enough injectives by taking products of towers

$$k_*E : \quad \cdots = E = E \to 0 \to 0 \cdots \to 0$$

with E injective. All the maps in k_*E (and hence in the product towers) are onto, so $\varprojlim^1$ vanishes on these injective towers. $\square$

Remark If we replace **Ab** by $\mathcal{A} =$ **mod-**R, **Ch**(**mod-**R) or any abelian category $\mathcal{A}$ satisfying Grothendieck's axiom ($AB5^*$) (filtered limits are exact), the above proof goes through to show that $\varprojlim^1 = R^1(\varprojlim)$ and $R^n(\varprojlim) = 0$ for $n \neq 0, 1$ as functors on the category of towers in $\mathcal{A}$. However, the proof breaks down for other abelian categories. Neeman has given examples of abelian categories with ($AB4^*$) in which Lemma 3.5.3 and Corollary 3.5.4 both fail; see *Invent. Math.* 148 (2002), 397-420.

Example 3.5.5 Set $A_0 = \mathbf{Z}$ and let $A_i = p^i\mathbf{Z}$ be the subgroup generated by p^i. Applying $\varprojlim$ to the short exact sequence of towers

$$0 \to \{p^i\mathbf{Z}\} \to \{\mathbf{Z}\} \to \left\{ \mathbf{Z} / p^i\mathbf{Z} \right\} \to 0$$

with p prime yields the uncountable group

$$\varprojlim^1 \{p^i\mathbf{Z}\} \cong \widehat{\mathbf{Z}}_p / \mathbf{Z}.$$

Here $\widehat{\mathbf{Z}}_p = \varprojlim \mathbf{Z}/p^i\mathbf{Z}$ is the group of p-adic integers.

Exercise 3.5.1 Let $\{A_i\}$ be a tower in which the maps $A_{i+1} \to A_i$ are inclusions. We may regard $A = A_0$ as a topological group in which the sets $a + A_i$ ($a \in A$, $i \geq 0$) are the open sets. Show that $\varprojlim A_i = \cap A_i$ is zero iff A is *Hausdorff*. Then show that $\varprojlim^1 A_i = 0$ iff A is *complete* in the sense that every Cauchy sequence has a limit, not necessarily unique. *Hint*: Show that A is complete and Hausdorff iff $A \cong \varprojlim \left(A / A_i \right)$.

To be explicit, A is Hausdorff if for all $\alpha, \beta \in A$, there exist open sets U, V (unions of $\{a + A_i\}_{a,i}$) with $\alpha \in U$, $\beta \in V$ and $U \cap V = \emptyset$. We first show that $\bigcap A_i = 0$ if and only if A is Hausdorff.

Suppose $\bigcap A_i = 0$. Let $\alpha, \beta \in A$ be distinct. As $\bigcap A_i = 0$ and $\alpha - \beta \neq 0$, we can choose a group A_i such that $\alpha - \beta \notin A_i$. Now observe that $\alpha \in \alpha + A_i$ and $\beta \in \beta + A_i$ trivially by construction, and that $(\alpha + A_i) \cap (\beta + A_i)$ must be the empty set, since $\alpha + A_i$ and $\beta + A_i$ are distinct cosets as $\alpha - \beta \neq 0$. Thus, A is Hausdorff.

Conversely, suppose A is Hausdorff. Let $\alpha \in A \setminus \{0\}$. By Hausdorff-ness, we can separate α from 0; i.e., there exists an open set $U \subseteq A$ such that $0 \in U$ but $\alpha \notin U$. Since U is open, there exists some $a + A_i \subseteq U$ such that $0 \in a + A_i \subseteq U$. This means that the coset $a + A_i$ is A_i. Hence, given an arbitrary $\alpha \in A$, there exists some A_i such that $\alpha \notin A_i$, and thus $\bigcap A_i = 0$.

We now turn to showing that A is complete and Hausdorff if and only if $A \cong \varprojlim \left(A / A_i \right)$.

To be explicit, A is complete if every Cauchy sequence converges, and a Cauchy sequence is a sequence $(a_n)_n$, $a_n \in A$, such that for all i, there exists $N = N(i)$ such that for all $j, k \geq N$, $a_j - a_k \in A_i$.

Consider the short exact sequence of towers

$$0 \to \{A_i\} \to \{A\} \to \left\{ {}^{A}\!/\!_{A_i} \right\} \to 0.$$

The derived functor $\varprojlim^n$ gives rise to a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \varprojlim A_i \arrow[r] & \varprojlim A \arrow[r] & \varprojlim {}^{A}\!/\!_{A_i} \arrow[dlll, "\delta"] \\ \ddots & & & \end{tikzcd}$$

As $\{A\}$ has identity maps, $\varprojlim A = A$. Furthermore, A is Hausdorff if and only if, by the first part of this exercise, $\varprojlim A_i = 0$. Thus we have

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & A \arrow[r] & \varprojlim {}^{A}\!/\!_{A_i} \arrow[dlll, "\delta"] \\ \ddots & & & \end{tikzcd}$$

It only remains to show that from this point, $A \cong \varprojlim {}^{A}\!/\!_{A_i}$ if and only if A is complete, for then we have an isomorphism $A \cong \varprojlim {}^{A}\!/\!_{A_i}$ if and only if A is both Hausdorff and complete, as we need to show.

So we proceed; consider a Cauchy sequence (a_n) in A. For every i, choose N_i such that for all $j, k \geq N_i$, $a_j - a_k \in A_i$. This implies that for all $j, k \geq N_i$, $a_j \equiv a_k \pmod{A_i}$. This means the maps (defined for any i) $\varphi_i\big((a_n)\big) = a_{N_i} \pmod{A_i}$ are well-defined maps $A \to {}^{A}\!/\!_{A_i}$ for each i, and by the universal property of directed limits, we get a map $\varphi : \varprojlim \{A\} = A \to \varprojlim \left\{ {}^{A}\!/\!_{A_i} \right\}$. If $\varphi\big((a_n)\big) = \varphi\big((b_n)\big)$, then we say (a_n) and (b_n) are equivalent Cauchy sequences. If $(b_n) \in \varprojlim \left\{ {}^{A}\!/\!_{A_i} \right\}$, then there exists $(a_n) \in \{A\}$ such that $\varphi\big((a_n)\big) = (b_n)$, because we may choose a_n to be a lift of b_n in A, and the sequence (a_n) is still Cauchy. Thus, $\varprojlim {}^{A}\!/\!_{A_i}$ is the completion of A, as, by defintion, the completion is the space of Cauchy sequences modulo equivalent Cauchy sequences. Finally, we return to the long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & \varprojlim A \arrow[r] & \varprojlim A/A_i \arrow[dlll, rounded corners, "\delta"] \\
\ddots & & &
\end{tikzcd}$$

The map $\varprojlim A \to \varprojlim A/A_i$ in this sequence is given by $(a) \mapsto ([a])$, so the image is all Cauchy sequences which are equivalent to a constant sequence (which converge by basic topological results under the Hausdorff assumption). It is injective by the diagram, and hence, this map is an isomorphism if and only if *all* Cauchy sequences converge, i.e., A is itself complete.

Defintion 3.5.6 A tower $\{A_i\}$ of abelian groups satisfies the *Mittag-Leffler condition* if for each k there exists a $j \geq k$ such that the image of $A_i \to A_k$ equals the image of $A_j \to A_k$ for all $i \geq j$. (The images of the A_i in A_k satisfy the *descending chain condition*.) For example, the Mittag-Leffler condition is satisfied if all the maps $A_{i+1} \to A_i$ in the tower $\{A_i\}$ are onto. We say that $\{A_i\}$ satisfies the *trivial* Mittag-Leffler condition if for each k there exists a $j > k$ such that the map $A_j \to A_k$ is zero.

Proposition 3.5.7 *If $\{A_i\}$ satisfies the Mittag-Leffler condition, then*

$$\varprojlim{}^1 A_i = 0.$$

Proof. If $\{A_i\}$ satisfies the trivial Mittag-Leffler condition, and $b_i \in A_i$ are given, set $a_k = b_k + \overline{b_{k+1}} + \cdots + \overline{b_{j-1}}$, where $\overline{b_i}$ denotes the image of b_i in A_k. (Note that $\overline{b_i} = 0$ for $i \geq j$.) Then Δ maps $(\cdots, a_1, a_0)$ to $(\cdots, b_1, b_0)$. Thus Δ is onto and $\varprojlim^1 A_i = 0$ when $\{A_i\}$ satisfies the trivial Mittag-Leffler condition. In the general case, let $B_k \subseteq A_k$ be the image of $A_i \to A_k$ for large i. The maps $B_{k+1} \to B_k$ are all onto, so $\varprojlim^1 B_k = 0$. The tower $\left\{A_k/B_k\right\}$ satisfies the trivial Mittag-Leffler condition, so $\varprojlim^1 A_k/B_k = 0$. From the short exact sequence

$$0 \to \{B_i\} \to \{A_i\} \to \left\{A_i/B_i\right\} \to 0$$

of towers, we see that $\varprojlim^1 A_i = 0$ as claimed. □

Exercise 3.5.2 Show that $\varprojlim^1 A_i = 0$ if $\{A_i\}$ is a tower of finite abelian groups, or a tower of finite-dimensional vector spaces over a field.

The following formula presages the Universal Coefficient theorems of the next section, as well as the spectral sequences of Chapter 5.

Theorem 3.5.8 *Let $\cdots \to C_1 \to C_0$ be a tower of chain complexes of abelian groups satisfying the Mittag-Leffler condition, and set $C = \varprojlim C_i$. Then there is an exact sequence for each q:*

$$0 \to \varprojlim{}^1 H_{q+1}(C_i) \to H_q(C) \to \varprojlim H_q(C_i) \to 0.$$

Proof. Let $B_i \subseteq Z_i \subseteq C_i$ be the subcomplexes of boundaries and cycles in the complex C_i, so that Z_i/B_i is the chain complex $H_*(C_i)$ with zero differentials. Applying the left exact functor $\varprojlim$ to $0 \to \{Z_i\} \to \{C_i\} \xrightarrow{d}$

$\{C_i[-1]\}$ shows that in fact $\varprojlim Z_i$ is the subcomplex Z of cycles in C. (The $[-1]$ refers to the supressed subscript on the chain complexes.) Let B denote the subcomplex $d(C)[1] = \left(C/Z\right)[1]$ of boundaries in C, so that Z/B is the chain complex $H_*(C)$ with zero differentials. From the exact sequence of towers

$$0 \to \{Z_i\} \to \{C_i\} \xrightarrow{d} \{B_i[-1]\} \to 0$$

we see that $\varprojlim^1 B_i = (\varprojlim^1 B_i[-1])[+1] = 0$ and that

$$0 \to B[-1] \to \varprojlim B_i[-1] \to \varprojlim^1 Z_i \to 0$$

is exact. From the exact sequence of towers

$$0 \to \{B_i\} \to \{Z_i\} \to H_*(C_i) \to 0$$

we see that $\varprojlim^1 Z_i \cong \varprojlim^1 H_*(C_i)$ and that

$$0 \to \varprojlim B_i \to Z \to \varprojlim H_*(C_i) \to 0$$

is exact. Hence C has the filtration by subcomplexes

$$0 \subseteq B \subseteq \varprojlim B_i \subseteq Z \subseteq C$$

whose filtration quotients are B, $\varprojlim^1 H_*(C_i)[1]$, $\varprojlim H_*(C_i)$, and C/Z respectively. The theorem follows, since $Z/B = H_*(C)$. $\square$

Variant If $\cdots \to C_1 \to C_0$ is a tower of cochain complexes satisfying the Mittag-Leffler condition, the sequence becomes

$$0 \to \varprojlim^1 H^{q-1}(C_i) \to H^q(C) \to \varprojlim H^q(C_i) \to 0.$$

Application 3.5.9 Let $H^*(X)$ denote the integral cohomology of a topological CW complex X. If $\{X_i\}$ is an increasing sequence of subcomplexes with $X = \cup X_i$, there is an exact sequence

$$0 \to \varprojlim^1 H^{q-1}(X_i) \to H^q(X) \to \varprojlim H^q(X_i) \to 0 \qquad (*)$$

for each q. This use of $\varprojlim^1$ to perform calculations in algebraic topology was discovered by Milnor in 1960 [Milnor] and thrust $\varprojlim^1$ into the limelight.

To derive this formula, let C_i denote the chain complex $\mathrm{Hom}(S(X_i), \mathbf{Z})$ used to compute $H^*(X_i)$. Since the inclusion $S(X_i) \subseteq S(X_{i+1})$ splits (because each $S_n(X_{i+1})/S_n(X_i)$ is a free abelian group), the maps $C_{i+1} \to C_i$ are onto, and the tower satisfies the Mittag-Leffler condition. Since X has the weak topology, $S(X)$ is the union of the $S(X_i)$, and therefore $H^*(X)$ is the cohomology of the cochain complex

$$\mathrm{Hom}(\cup S(X_i), \mathbf{Z}) = \varprojlim \mathrm{Hom}(S(X_i), \mathbf{Z}) = \varprojlim C_i.$$

A historical remark: Milnor proved that the sequence $(*)$ is also valid if H^* is replaced by any generalized cohomology theory, such as topological K-theory.

Application 3.5.10 Let A be an R-module that is the union of submodules $\cdots \subseteq A_i \subseteq A_{i+1} \subseteq \cdots$. Then for every R-module B and every q the sequence

$$0 \to \varprojlim^1 \mathrm{Ext}_R^{q-1}(A_i, B) \to \mathrm{Ext}_R^q(A, B) \to \varprojlim \mathrm{Ext}_R^q(A_i, B) \to 0$$

is exact. For $\mathbf{Z}_{p^\infty} = \cup \mathbf{Z}/p^i$, this gives a short exact sequence for every B:

$$0 \to \varprojlim^1 \mathrm{Hom}\left(\mathbf{Z}/p^i, B\right) \to \mathrm{Ext}^1_{\mathbf{Z}}\left(\mathbf{Z}_{p^\infty}, B\right) \to \widehat{B}_p \to 0,$$

where the group $\widehat{B}_p = \varprojlim \left({}^{B}\!/\!_{p^i B} \right)$ is the p-adic completion of B. This generalizes the calculation $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}_{p^\infty}, \mathbf{Z}) \cong \widehat{\mathbf{Z}}_p$ of 3.3.3. To see this, let E be a fixed injective resolution of B, and consider the tower of cochain complexes

$$\mathrm{Hom}(A_{i+1}, E) \to \mathrm{Hom}(A_i, E) \to \cdots \to \mathrm{Hom}(A_0, E).$$

Each $\mathrm{Hom}(-, E_n)$ is contravariant exact, so each map in the tower is a surjection. The cohomology of $\mathrm{Hom}(A_i, E)$ is $\mathrm{Ext}^*(A_i, B)$, and $\mathrm{Ext}^*(A, B)$ is the cohomology of

$$\mathrm{Hom}(\cup A_i, E) = \varprojlim \mathrm{Hom}(A_i, E).$$

Exercise 3.5.3 Show that $\mathrm{Ext}^1_{\mathbf{Z}}\left(\mathbf{Z}\left[\frac{1}{p}\right], \mathbf{Z}\right) \cong {}^{\widehat{\mathbf{Z}}_p}\!/\!_{\mathbf{Z}}$ using $\mathbf{Z}\left[\frac{1}{p}\right] = \cup p^{-i}\mathbf{Z}$; cf. exercise 3.3.1. Then show that $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Q}, B) = {}^{(\prod_p \widehat{B}_p)}\!/\!_{B}$ for torsionfree B.

By Application 3.5.10 above, since $\mathbf{Z}\left[\frac{1}{p}\right] = \bigcup p^{-i}\mathbf{Z}$, for all B and q we have the short exact sequence

$$0 \to \varprojlim{}^1 \mathrm{Ext}^{q-1}_{\mathbf{Z}}\left(p^{-i}\mathbf{Z}, B\right) \to \mathrm{Ext}^q_{\mathbf{Z}}\left(\mathbf{Z}\left[\tfrac{1}{p}\right], B\right) \to \varprojlim \mathrm{Ext}^q_{\mathbf{Z}}\left(p^{-i}\mathbf{Z}, B\right) \to 0.$$

Choose $B = \mathbf{Z}$ and $q = 1$. We first claim that

$$\varprojlim{}^1 \mathrm{Ext}^0_{\mathbf{Z}}\left(p^{-i}\mathbf{Z}, \mathbf{Z}\right) = \varprojlim{}^1 \mathrm{Hom}_{\mathbf{Z}}\left(p^{-i}\mathbf{Z}, \mathbf{Z}\right) \cong \varprojlim{}^1 p^i\mathbf{Z} \cong {}^{\widehat{\mathbf{Z}}_p}\!/\!_{\mathbf{Z}}.$$

Indeed, $f \in \mathrm{Hom}(p^{-i}\mathbf{Z}, \mathbf{Z})$ is determined by the image of the generator $\frac{1}{p^i}$ in $\mathbf{Z}$, so $\mathrm{Hom}(p^{-i}\mathbf{Z}, \mathbf{Z})$ is infinite cyclic. Observe that the tower maps $p^{-(i+1)}\mathbf{Z} \overset{p}{\hookrightarrow} p^{-i}\mathbf{Z}$ functorially yield maps $\mathrm{Hom}(p^{-i}\mathbf{Z}, \mathbf{Z}) \xrightarrow{p^*} \mathrm{Hom}(p^{-(i+1)}\mathbf{Z}, \mathbf{Z})$, that is, $p^i\mathbf{Z} \xrightarrow{p} p^{i+1}\mathbf{Z}$, and then we show $\varprojlim{}^1 p^i\mathbf{Z}$ must be ${}^{\widehat{\mathbf{Z}}_p}\!/\!_{\mathbf{Z}}$. To see this, consider the short exact sequence of towers

$$0 \to \{p^i\mathbf{Z}\} \to \{\mathbf{Z}\} \to \left\{{}^{\mathbf{Z}}\!/\!_{p^i\mathbf{Z}}\right\} \to 0$$

which has $\varprojlim{}^n$ long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim p^i\mathbf{Z} \arrow[r] & \varprojlim \mathbf{Z} \arrow[r] & \varprojlim {}^{\mathbf{Z}}\!/\!_{p^i\mathbf{Z}} \arrow[dll, bend left, "\delta"] \\
& \varprojlim{}^1 p^i\mathbf{Z} \arrow[r] & \varprojlim{}^1 \mathbf{Z} \arrow[r] & \cdots .
\end{tikzcd}$$

As the tower $\{\mathbf{Z}\}$ has identity maps, which are onto, by Lemma 3.5.3, $\varprojlim{}^1 \mathbf{Z} = 0$, and therefore $\varprojlim{}^1 p^i\mathbf{Z} = \mathrm{coker}\left(\varprojlim \mathbf{Z} \to \varprojlim {}^{\mathbf{Z}}\!/\!_{p^i\mathbf{Z}}\right)$. Observe that $\varprojlim \mathbf{Z} \cong \mathbf{Z}$, and that $\varprojlim {}^{\mathbf{Z}}\!/\!_{p^i\mathbf{Z}} \cong \widehat{\mathbf{Z}}_p$ by

Example 3.3.3. Therefore, $\varprojlim^1 p^i\mathbf{Z} \cong \widehat{\mathbf{Z}}_p/\mathbf{Z}$, as claimed.

Second, we claim that

$$\varprojlim \operatorname{Ext}^1_{\mathbf{Z}}\left(p^{-i}\mathbf{Z}, \mathbf{Z}\right) = \varprojlim 0 = 0.$$

To see this, it is enough to show that $p^{-i}\mathbf{Z}$ is projective. Indeed, let $M \to N$ be a surjection and let $f : p^{-i}\mathbf{Z} \to N$. The map f is determined by the image of the generator $\frac{1}{p^i}$ in N; call it n. Lift n to a preimage $m \in M$, and then the map $p^{-i}\mathbf{Z} \to M$ defined by $\frac{1}{p^i} \mapsto m$ causes the following diagram to commute:

$$\begin{tikzcd}
 & & p^{-i}\mathbf{Z} \arrow[ddll, dashed, bend right] \arrow[dd] & \frac{1}{p^i} \arrow[dll, dashed, mapsto, bend right] \arrow[d, mapsto, "f"] & \\
 & m \arrow[rr, mapsto] & & n & \\
M \arrow[rr] & & M \arrow[rr] & & 0.
\end{tikzcd}$$

Hence, $\varprojlim \operatorname{Ext}^1(p^{-i}\mathbf{Z}, \mathbf{Z}) = \varprojlim 0 = 0$, as desired. Critically, note the independence of $B = \mathbf{Z}$ from the above justification; thus, it is the case that

$$\varprojlim \operatorname{Ext}^1_{\mathbf{Z}}\left(p^{-i}\mathbf{Z}, B\right) = \varprojlim 0 = 0 \tag{$\star$}$$

for all B, a fact we will return to later in the exercise. Regardless, the initial short exact sequence simplifies to

$$0 \to \widehat{\mathbf{Z}}_p/\mathbf{Z} \to \operatorname{Ext}^1_{\mathbf{Z}}\left(\mathbf{Z}\left[\tfrac{1}{p}\right], \mathbf{Z}\right) \to 0 \to 0,$$

and $\operatorname{Ext}\left(\mathbf{Z}\left[\frac{1}{p}\right], \mathbf{Z}\right) \cong \widehat{\mathbf{Z}}_p/\mathbf{Z}$, as we wished to show.

• • •

Next, we need to show that $\operatorname{Ext}^1(\mathbf{Q}, B) \cong \left(\prod_p \widehat{B}_p\right)/B$ if B is torsionfree. Write $P_j = p_1 \cdots p_j$ for the product of the first j primes, and then observe that $\mathbf{Q} = \bigcup_j \mathbf{Z}\left[\frac{1}{P_j}\right]$. We thus have the following short exact sequence for all B and q by Application 3.5.10:

$$0 \to \varprojlim^1 \operatorname{Ext}^{q-1}_{\mathbf{Z}}\left(\mathbf{Z}\left[\tfrac{1}{P_j}\right], B\right) \to \operatorname{Ext}^q_{\mathbf{Z}}(\mathbf{Q}, B) \to \varprojlim \operatorname{Ext}^q_{\mathbf{Z}}\left(\mathbf{Z}\left[\tfrac{1}{P_j}\right], B\right) \to 0.$$

Choose $q = 1$.

We first claim that since $\mathbf{Z}\left[\frac{1}{P_j}\right] = \bigcup_i {P_j}^{-i}\mathbf{Z}$, we may use the first part of this exercise to conclude

$$\varprojlim \operatorname{Ext}^1_{\mathbf{Z}}\left(\mathbf{Z}\left[\tfrac{1}{P_j}\right], B\right) \cong \varprojlim \widehat{B}_{P_j}\big/B.$$

To see this, take the short exact sequence

$$0 \to \varprojlim{}^1 \operatorname{Ext}^0_{\mathbf{Z}}\left({P_j}^{-i}\mathbf{Z}, B\right) \to \operatorname{Ext}^1_{\mathbf{Z}}\left(\mathbf{Z}\left[\tfrac{1}{P_j}\right], B\right) \to \varprojlim \operatorname{Ext}^1_{\mathbf{Z}}\left({P_j}^{-i}\mathbf{Z}, B\right) \to 0.$$

By $(\star)$, the third term $\varprojlim \operatorname{Ext}^1({P_j}^{-i}\mathbf{Z}, B)$ is 0. The first term,

$$\varprojlim{}^1 \operatorname{Ext}^0({P_j}^{-i}\mathbf{Z}, B) = \varprojlim{}^1 \operatorname{Hom}({P_j}^{-i}\mathbf{Z}, B),$$

is $\varprojlim^1 {P_j}^i B$, since a map f is determined by the image of the generator $\frac{1}{{P_j}^i}$ in B, and $\varprojlim^1 {P_j}^i B$ is $\widehat{B}_{P_j}\big/B$, since given a short exact sequence of towers $0 \to \{{P_j}^i B\} \to \{B\} \to \left\{B\big/{P_j}^i B\right\} \to 0$ and noting that $\varprojlim^1 B = 0$ by Lemma 3.5.3, we again have a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim {P_j}^i B \arrow[r] & \varprojlim B \arrow[r] & \varprojlim B\big/{P_j}^i B \arrow[dll, "\delta"] \\
& \varprojlim^1 {P_j}^i B \arrow[r] & 0, &
\end{tikzcd}$$

and therefore

$$\varprojlim{}^1 {P_j}^i B \cong \operatorname{coker}\left(\varprojlim B \to \varprojlim B\big/{P_j}^i B\right) \cong \operatorname{coker}\left(B \to \widehat{B}_{P_j}\right) \cong \widehat{B}_{P_j}\big/B,$$

as claimed. Therefore our short exact sequence is $0 \to \widehat{B}_{P_j}\big/B \to \operatorname{Ext}^1\left(\mathbf{Z}\left[\frac{1}{P_j}\right], B\right) \to 0 \to 0$, so $\varprojlim \operatorname{Ext}^1\left(\mathbf{Z}\left[\frac{1}{P_j}\right], B\right) \cong \varprojlim \widehat{B}_{P_j}\big/B$, as claimed. Now, computing $\varprojlim \widehat{B}_{P_j}\big/B$, observe that

$$\varprojlim \widehat{B}_{P_j}\big/B = \varprojlim \widehat{B}_{p_1\cdots p_j}\big/B = \varprojlim \left(\prod_{\substack{p_k \\ 1\le k\le j}} \widehat{B}_{p_k}\right)\Big/B = \left(\prod_{\substack{p \\ \text{prime}}} \widehat{B}_p\right)\Big/B.$$

It only remains to be seen that $\varprojlim^1 \operatorname{Ext}^0\left(\mathbf{Z}\left[\frac{1}{P_j}\right], B\right) = 0$, for then the short exact sequence

$$0 \to \varprojlim{}^1 \operatorname{Ext}^0_{\mathbf{Z}}\left(\mathbf{Z}\left[\tfrac{1}{P_j}\right], B\right) \to \operatorname{Ext}^1_{\mathbf{Z}}(\mathbf{Q}, B) \to \varprojlim \operatorname{Ext}^1_{\mathbf{Z}}\left(\mathbf{Z}\left[\tfrac{1}{P_j}\right], B\right) \to 0$$

simplifies to

$$0 \to 0 \to \operatorname{Ext}^1_{\mathbf{Z}}(\mathbf{Q}, B) \to \left(\prod_p \widehat{B}_p\right)\!\big/ B \to 0,$$

so $\operatorname{Ext}^1(\mathbf{Q}, B) \cong \left(\prod_p \widehat{B}_p\right)\!\big/B$, as we wish to show. To see that $\varprojlim^1 \operatorname{Ext}^0\left(\mathbf{Z}\left[\frac{1}{P_j}\right], B\right) = \varprojlim^1 \operatorname{Hom}\left(\mathbf{Z}\left[\frac{1}{P_j}\right], B\right) = 0$, we claim the tower satisfies the Mittag-Leffler condition, so that by Proposition 3.5.7, $\varprojlim^1 \operatorname{Hom}\left(\mathbf{Z}\left[\frac{1}{P_j}\right], B\right) = 0$ as desired. To prove this claim and complete the exercise, fix an arbitrary k; we must show there exists $j \geq k$ such that

$$\operatorname{im}\left(\operatorname{Hom}\left(\mathbf{Z}\left[\frac{1}{P_i}\right], B\right) \to \operatorname{Hom}\left(\mathbf{Z}\left[\frac{1}{P_k}\right], B\right)\right) = \operatorname{im}\left(\operatorname{Hom}\left(\mathbf{Z}\left[\frac{1}{P_j}\right], B\right) \to \operatorname{Hom}\left(\mathbf{Z}\left[\frac{1}{P_k}\right], B\right)\right)$$

for all $i \geq j$. Indeed, such a j is $j = k+1$. Since $P_k = p_1 \cdots p_k$ divides $P_{k+1} = p_1 \cdots p_k \cdot p_{k+1}$, the map $\mathbf{Z}\left[\frac{1}{P_k}\right] \to \mathbf{Z}\left[\frac{1}{P_{k+1}}\right]$ is multiplication by $\frac{1}{p_{k+1}}$. Thus the map $\operatorname{Hom}\left(\mathbf{Z}\left[\frac{1}{P_{k+1}}\right], B\right) \to \operatorname{Hom}\left(\mathbf{Z}\left[\frac{1}{P_k}\right], B\right)$ is induced by multiplication by $\frac{1}{p_{k+1}}$. Let $i \geq j = k+1$. Observe that the image of $\operatorname{Hom}\left(\mathbf{Z}\left[\frac{1}{P_{k+1}}\right], B\right) \to \operatorname{Hom}\left(\mathbf{Z}\left[\frac{1}{P_k}\right], B\right)$ must equal the image of $\operatorname{Hom}\left(\mathbf{Z}\left[\frac{1}{P_i}\right], B\right) \to \operatorname{Hom}\left(\mathbf{Z}\left[\frac{1}{P_k}\right], B\right)$, because

$$\begin{aligned}
&\operatorname{im}\left(\operatorname{Hom}\left(\mathbf{Z}\left[\tfrac{1}{P_{k+1}}\right], B\right) \to \operatorname{Hom}\left(\mathbf{Z}\left[\tfrac{1}{P_k}\right], B\right)\right) \\
&= \left\{ f : \mathbf{Z}\left[\tfrac{1}{P_k}\right] \to B \mid f = g\tfrac{1}{p_{k+1}}^* \text{ where } \mathbf{Z}\left[\tfrac{1}{P_k}\right] \xrightarrow{\frac{1}{p_{k+1}}^*} \mathbf{Z}\left[\tfrac{1}{P_{k+1}}\right] \xrightarrow{g} B \right\}, \text{ and} \\
&\operatorname{im}\left(\operatorname{Hom}\left(\mathbf{Z}\left[\tfrac{1}{P_i}\right], B\right) \to \operatorname{Hom}\left(\mathbf{Z}\left[\tfrac{1}{P_k}\right], B\right)\right) \\
&= \left\{ f : \mathbf{Z}\left[\tfrac{1}{P_k}\right] \to B \mid f = h\tfrac{1}{p_{k+1}\cdots p_i}^* \text{ where } \mathbf{Z}\left[\tfrac{1}{P_k}\right] \xrightarrow{\frac{1}{p_{k+1}\cdots p_i}^*} \mathbf{Z}\left[\tfrac{1}{P_i}\right] \xrightarrow{h} B \right\} \\
&= \left\{ f : \mathbf{Z}\left[\tfrac{1}{P_k}\right] \to B \mid f = h\tfrac{1}{p_{k+2}\cdots p_i}^* \tfrac{1}{p_{k+1}}^* \text{ where } \mathbf{Z}\left[\tfrac{1}{P_k}\right] \xrightarrow{\frac{1}{p_{k+1}}^*} \mathbf{Z}\left[\tfrac{1}{P_{k+1}}\right] \xrightarrow{\frac{1}{p_{k+2}\cdots p_i}^*} \mathbf{Z}\left[\tfrac{1}{P_i}\right] \xrightarrow{h} B \right\},
\end{aligned}$$

so clearly if we let $g = h\frac{1}{p_{k+2}\cdots p_i}^*$, then

$$\begin{aligned}
&f : \mathbf{Z}\left[\frac{1}{P_k}\right] \xrightarrow{\frac{1}{p_{k+1}}^*} \mathbf{Z}\left[\frac{1}{P_{k+1}}\right] \xrightarrow{g} B \\
=&f : \mathbf{Z}\left[\frac{1}{P_k}\right] \xrightarrow{\frac{1}{p_{k+1}}^*} \mathbf{Z}\left[\frac{1}{P_{k+1}}\right] \xrightarrow{\frac{1}{p_{k+2}\cdots p_i}^*} \mathbf{Z}\left[\frac{1}{P_i}\right] \xrightarrow{h} B,
\end{aligned}$$

so

$$\operatorname{im}\left(\operatorname{Hom}\left(\mathbf{Z}\left[\frac{1}{P_i}\right], B\right) \to \operatorname{Hom}\left(\mathbf{Z}\left[\frac{1}{P_k}\right], B\right)\right) \subseteq \operatorname{im}\left(\operatorname{Hom}\left(\mathbf{Z}\left[\frac{1}{P_{k+1}}\right], B\right) \to \operatorname{Hom}\left(\mathbf{Z}\left[\frac{1}{P_k}\right], B\right)\right),$$

and for the other inclusion, if we let $h = g\,(p_{k+2} \cdots p_i)^*$, then

$$\begin{aligned}
&f : \mathbf{Z}\left[\frac{1}{P_k}\right] \xrightarrow{\frac{1}{p_{k+1}}^*} \mathbf{Z}\left[\frac{1}{P_{k+1}}\right] \xrightarrow{\frac{1}{p_{k+2}\cdots p_i}^*} \mathbf{Z}\left[\frac{1}{P_i}\right] \xrightarrow{h} B \\
=&f : \mathbf{Z}\left[\frac{1}{P_k}\right] \xrightarrow{\frac{1}{p_{k+1}}^*} \mathbf{Z}\left[\frac{1}{P_{k+1}}\right] \xrightarrow{\frac{1}{p_{k+2}\cdots p_i}^*} \mathbf{Z}\left[\frac{1}{P_i}\right] \xrightarrow{(p_{k+2}\cdots p_i)^*} \mathbf{Z}\left[\frac{1}{P_{k+1}}\right] \xrightarrow{g} B \\
=&f : \mathbf{Z}\left[\frac{1}{P_k}\right] \xrightarrow{\frac{1}{p_{k+1}}^*} \mathbf{Z}\left[\frac{1}{P_{k+1}}\right] \xrightarrow{g} B,
\end{aligned}$$

so

$$\operatorname{im}\left(\operatorname{Hom}\left(\mathbf{Z}\left[\frac{1}{P_{k+1}}\right], B\right) \to \operatorname{Hom}\left(\mathbf{Z}\left[\frac{1}{P_k}\right], B\right)\right) \subseteq \operatorname{im}\left(\operatorname{Hom}\left(\mathbf{Z}\left[\frac{1}{P_i}\right], B\right) \to \operatorname{Hom}\left(\mathbf{Z}\left[\frac{1}{P_k}\right], B\right)\right),$$

and therefore the tower satisfies the Mittag-Leffler condition, as desired.

Application 3.5.11 Let $C = C_{**}$ be a double chain complex, viewed as a lattice in the plane, and let T_nC be the quotient double complex obtained by brutally truncating C at the vertical line $p = -n$:

$$(T_nC)_{pq} = \begin{cases} C_{pq} & \text{if } p \geq -n \\ 0 & \text{if } p < -n \end{cases}.$$

Then $\operatorname{Tot}(C) = \operatorname{Tot}^{\prod}(C)$ is the inverse limit of the tower of surjections

$$\cdots \to \operatorname{Tot}(T_{i+1}C) \to \operatorname{Tot}(T_iC) \to \cdots \to \operatorname{Tot}(T_0C).$$

Therefore there is a short exact sequence for each q:

$$0 \to \varprojlim{}^1 H_{q+1}(\operatorname{Tot}(T_iC)) \to H_q(\operatorname{Tot}(C)) \to \varprojlim H_q(\operatorname{Tot}(T_iC)) \to 0.$$

This is especially useful when C is a second quadrant double complex, because the truncated complexes have only a finite number of nonzero columns.

Exercise 3.5.4 Let C be a second quadrant double complex with exact rows, and let B^h_{pq} be the image of $d^h : C_{pq} \to C_{p-1,q}$. Show that $H_{p+q}\operatorname{Tot}(T_{-p}C) \cong H_q(B^h_{p*}, d^v)$. Then let $b = d^h(a)$ be an element of B^h_{pq} representing a cycle ξ in $H_{p+q}\operatorname{Tot}(T_{-p}C)$ and show that the image of ξ in $H_{p+q}\operatorname{Tot}(T_{-p-1}C)$ is represented by $d^v(a) \in B^h_{p+1,q-1}$. This provides an effective method for calculating $H_*\operatorname{Tot}(C)$.

Vista 3.5.12 Let I be any poset and $\mathcal{A}$ any abelian category satisfying $(AB4^*)$. The following construction of the right derived functors of lim is taken from [Roos] and generalizes the construction of $\varprojlim^1$ in this section.

Given $A : I \to \mathcal{A}$, we define C_k to be the product over the set of all chains $i_k < \cdots < i_0$ in I of the objects A_{i_0}. Letting $pr_{i_k \cdots i_1}$ denote the projection of C_k onto the $(i_k < \cdots < i_1)^{st}$ factor and f_0 denote the map $A_{i_1} \to A_{i_0}$ associated to $i_1 < i_0$, we define $d^0 : C_{k-1} \to C_k$ to be the map whose $(i_k < \cdots < i_0)^{th}$ factor is $f_0(pr_{i_k \cdots i_1})$. For $1 \leq p \leq k$, we define $d^p : C_{k-1} \to C_k$ to be the map whose $(i_k < \cdots < i_0)^{th}$ factor is the projection onto the $(i_k < \cdots < \widehat{i_p} < \cdots < i_0)^{th}$ factor. This data defines a cochain complex C_*A whose differential $C_{k-1} \to C_k$ is the alternating sum $\sum_{p=0}^{k}(-1)^p d^p$, and we define $\lim^n_{i\in I} A$ to be $H^n(C_*A)$. (The data actually forms a *cosimplicial object* of $\mathcal{A}$; see Chapter 8.)

It is easy to see that $\lim^0_{i\in I} A$ is the limit $\lim_{i\in I} A$. An exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}^I$ gives rise to a short exact sequence $0 \to C_*A \to C_*B \to C_*C \to 0$ in $\mathcal{A}$, whence an exact sequence

$$0 \to \lim_{i\in I} A \to \lim_{i\in I} B \to \lim_{i\in I} C \to \lim_{i\in I}{}^1 A \to \lim_{i\in I}{}^1 B \to \lim_{i\in I}{}^1 C \to \lim_{i\in I}{}^2 A \to \cdots .$$

Therefore the functors $\{\lim^n_{i\in I}\}$ form a cohomological δ-functor. It turns out that they are universal when $\mathcal{A}$ has enough injectives, so in fact $R^n \lim_{i\in I} \cong \lim^n_{i\in I}$.

Remark Let $\aleph_d$ denote the d^{th} infinite cardinal number, $\aleph_0$ being the cardinality of $\{1, 2, \cdots\}$. If I is a directed poset of cardinality $\aleph_d$, or a filtered category with $\aleph_d$ morphisms, Mitchell proved in [Mitch] that $R^n \varprojlim$ vanishes for $n \geq d + 2$.

Exercise 3.5.5 (Pullback) Let $\to\leftarrow$ denote the poset $\{x, y, z\}$, $x < z$ and $y < z$, so that $\lim\limits_{\to\leftarrow} A_i$ is the pullback of A_x and A_y over A_z. Show that $\lim\limits_{\to\leftarrow}{}^1 A_i$ is the cokernel of the difference map $A_x \times A_y \to A_z$ and that $\lim\limits_{\to\leftarrow}{}^n = 0$ for $n \neq 0, 1$.

Given $I = \bullet \to \bullet \leftarrow \bullet$, write $\mathcal{A}^I$ as

$$\begin{tikzcd} & A_y \arrow[d, "g"] \\ A_x \arrow[r, "f"'] & A_z \end{tikzcd}$$

From Vista 3.5.12, we construct C_k for all k. See that $C_0 = A_x \times A_y \times A_z$ where the chains are x, y, and z, and $C_1 = A_z \times A_z$ where the chains are $x < z$ and $y < z$. Furthermore, $C_k = 0$ for $k \notin \{0, 1\}$, since there are no longer chains. Thus the only nontrivial differential is $d : C_0 \to C_1$. By definition, $d = \sum\limits_{p=0}^{1}(-1)^p d^p = d^0 - d^1$, where $d^0 : C_0 \to C_1$ is the map $d^0(a_x, a_y, a_z) = (f(a_x), g(a_y))$ and $d^1(a_x, a_y, a_z) = (a_z, a_z)$. Therefore,

$$d(a_x, a_y, a_z) = \big(f(a_x) - a_z, g(a_y) - a_z\big).$$

Observe that

$$\begin{aligned}
\lim_{\to\leftarrow} A_i = H^0(C_*) &= {}^{\ker d}\!/_{\mathrm{im}(C_{-1} \to C_0)} = {}^{\ker d}\!/_{0} \\
&\cong \ker d = \{(a_x, a_y, a_z) \in A_x \times A_y \times A_z \mid f(a_x) = a_z = g(a_y)\} \\
&\cong \{(a_x, a_y) \in A_x \times A_y \mid f(a_x) = g(a_y)\} \\
&= A_x \times_{A_z} A_y,
\end{aligned}$$

where $A_x \times_{A_z} A_y$ denotes the pullback

$$\begin{tikzcd}
A_x \times_{A_z} A_y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & A_y \arrow[d, "g"] \\
A_x \arrow[r, "f"'] & A_z
\end{tikzcd}$$

as we were asked to show. Furthermore,

$$\lim_{\to\leftarrow}{}^1 A_i = H^1(C_*) = {}^{\ker(C_1 \to C_2)}\!\diagup_{\operatorname{im} d} = {}^{C_1}\!\diagup_{\operatorname{im} d} = \operatorname{coker} d,$$

and we claim $\operatorname{coker} d \cong \operatorname{coker} \mathit{diff}$, where $\mathit{diff} : A_x \times A_y \to A_z$, $\mathit{diff}(a_x, a_y) = f(a_x) - g(a_y)$ is the difference map. To prove the claim, we show that

$$\begin{aligned}
\operatorname{coker} d &= {}^{C_1}\!\diagup_{\operatorname{im} d} \\
&= {}^{A_z \times A_z}\!\diagup_{\{(a,b) \mid a = f(a_x) - a_z, b = g(a_y) - a_z\}} \\
&\cong {}^{A_z}\!\diagup_{\{a_z \mid f(a_x) - g(a_y) = a_z\}} \\
&= \operatorname{coker} \mathit{diff}
\end{aligned}$$

using the map $\varphi : A_z \times A_z \to \operatorname{coker} \mathit{diff}$, $\varphi(a, b) = [a - b]$. This map

- is surjective, since for all $[a] \in \operatorname{coker} \mathit{diff}$, $\varphi(a, 0) = [a - 0] = [a]$,
- has kernel $\operatorname{im} d$, since if $(a, b) \in \ker \varphi$, then $\varphi(a, b) = [a - b] = [0]$, so $f(a_x) - g(a_y) = a - b$, hence $f(a_x) - g(a_y) + b = a$, and so

$$\begin{aligned}
d(a_x, a_y, g(a_y) - b) &= \Big(f(a_x) - \big(g(a_y) - b\big), g(a_y) - \big(g(a_y) - b\big)\Big) \\
&= \big(f(a_x) - g(a_y) + b, g(a_y) - g(a_y) + b\big) \\
&= (a, b),
\end{aligned}$$

  so $\ker \varphi \subseteq \operatorname{im} d$, and conversely,

$$\begin{aligned}
\varphi d(a_x, a_y, a_z) &= \varphi(f(a_x) - a_z, g(a_y) - a_z) \\
&= [f(a_x) - a_z - g(a_y) + a_z] \\
&= [f(a_x) - g(a_y)] = [0],
\end{aligned}$$

  so $\operatorname{im} d \subseteq \ker \varphi$.

Thus, by the first isomorphism theorem, $\operatorname{coker} \mathit{diff} \cong {A_z \times A_z}/{\ker \varphi} = {C_1}/{\operatorname{im} d} = \operatorname{coker} d = \underset{\to\leftarrow}{\lim}{}^1 A_i$, as requested. Finally,

$$\underset{\to\leftarrow}{\lim}{}^n A_i = H^n(C_*) = {\ker(C_n \to C_{n+1})}/{\operatorname{im}(C_{n-1} \to C_n)} = {0}/{0} = 0$$

for $n \notin \{0, 1\}$, as needed.

3.6 Universal Coefficient Theorem

There is a very useful formula for using the homology of a chain complex P to compute the homology of the complex $P \otimes M$. Here is the most useful general formulation we can give:

Theorem 3.6.1 (Künneth formula) *Let P be a chain complex of flat right R-modules such that each submodule $d(P_n)$ of P_{n-1} is also flat. Then for every n and every left R-module M, there is an exact sequence*

$$0 \to H_n(P) \otimes_R M \to H_n(P \otimes_R M) \to \operatorname{Tor}_1^R(H_{n-1}(P), M) \to 0.$$

Proof. The long exact Tor sequence associated to $0 \to Z_n \to P_n \to d(P_n) \to 0$ shows that each Z_n is also flat (exercise 3.2.2). Since $\operatorname{Tor}_1^R(d(P_n), M) = 0$,

$$0 \to Z_n \otimes M \to P_n \otimes M \to d(P_n) \otimes M \to 0$$

is exact for every n. These assemble to give a short exact sequence of chain complexes $0 \to Z \otimes M \to P \otimes M \to d(P) \otimes M \to 0$. Since the differentials in the Z and $d(P)$ complexes are zero, the homology sequence is

$$\begin{tikzcd}
H_{n+1}(dP \otimes M) \arrow[r, "\partial"] \arrow[d, equal, "\wr"] & H_n(Z \otimes M) \arrow[r] \arrow[d, equal, "\wr"] & H_n(P \otimes M) \arrow[r] & H_n(dP \otimes M) \arrow[r, "\partial"] \arrow[d, equal, "\wr"] & H_{n-1}(Z \otimes M) \arrow[d, equal, "\wr"] \\
d(P_{n+1}) \otimes M & Z_n \otimes M & & d(P_n) \otimes M & Z_{n-1} \otimes M.
\end{tikzcd}$$

Using the definition of ∂, it is immediate that $\partial = i \otimes M$, where i is the inclusion of $d(P_{n+1})$ in Z_n. On the other hand,

$$0 \to d(P_{n+1}) \xrightarrow{i} Z_n \to H_n(P) \to 0$$

is a flat resolution of $H_n(P)$, so $\operatorname{Tor}_*(H_n(P), M)$ is the homology of

$$0 \to d(P_{n+1}) \otimes M \xrightarrow{\partial} Z_n \otimes M \to 0.$$

□

Universal Coefficient Theorem for Homology 3.6.2 *Let P be a chain complex of free abelian groups. Then for every n and every abelian group M the Künneth formula 3.6.1 splits noncanonically, yielding a direct sum decomposition*

$$H_n(P \otimes M) \cong H_n(P) \otimes M \oplus \operatorname{Tor}_1^{\mathbf{Z}}(H_{n-1}(P), M).$$

Proof. We shall use the well-known fact that every subgroup of a free abelian group is free abelian [KapIAB, section 15]. Since $d(P_n)$ is a subgroup of P_{n+1}, it is free abelian. Hence the surjection $P_n \to d(P_n)$ splits, giving a noncanonical decomposition

$$P_n \cong Z_n \oplus d(P_n).$$

Applying $\otimes M$, we see that $Z_n \otimes M$ is a direct summand of $P_n \otimes M$; a fortiori, $Z_n \otimes M$ is a direct summand of the intermediate group

$$\ker(d_n \otimes 1 : P_n \otimes M \to P_{n-1} \otimes M).$$

Modding out $Z_n \otimes M$ and $\ker(d_n \otimes 1)$ by the common image of $d_{n+1} \otimes 1$, we see that $H_n(P) \otimes M$ is a direct summand of $H_n(P \otimes M)$. Since P and $d(P)$ are flat, the Künneth formula tells us that the other summand is $\mathrm{Tor}_1(H_{n-1}(P), M)$. □

Theorem 3.6.3 (Künneth formula for complexes) *Let P and Q be right and left R-module complexes, respectively. Recall from 2.7.1 that the* tensor product complex *$P \otimes_R Q$ is the complex whose degree n part is $\bigoplus_{p+q=n} P_p \otimes Q_q$ and whose differential is given by $d(a \otimes b) = (da) \otimes b + (-1)^p a \otimes (db)$ for $a \in P_p$, $b \in Q_q$. If P_n and $d(P_n)$ are flat for each n, then there is an exact sequence*

$$0 \to \bigoplus_{p+q=n} H_p(P) \otimes H_q(Q) \to H_n(P \otimes_R Q) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(P), H_q(Q)) \to 0$$

for each n. If $R = \mathbf{Z}$ and P is a complex of free abelian groups, this sequence is noncanonically split.

Proof. Modify the proof given in 3.6.1 for $Q = M$. □

Application 3.6.4 (Universal Coefficient Theorem in topology) Let $S(X)$ denote the singular chain complex of a topological space X; each $S_n(X)$ is a free abelian group. If M is any abelian group, the homology of X with "coefficients" in M is

$$H_*(X; M) = H_*(S(X) \otimes M).$$

Writing $H_*(X)$ for $H_*(X; \mathbf{Z})$, the formula in this case becomes

$$H_n(X; M) \cong H_n(X) \otimes M \oplus \mathrm{Tor}_1^{\mathbf{Z}}(H_{n-1}(X), M).$$

This formula is often called the Universal Coefficient Theorem in topology.

If Y is another topological space, the Eilenberg-Zilber theorem 8.5.1 (see [MacH, VIII.8]) states that $H_*(X \times Y)$ is the homology of the tensor product complex $S(X) \otimes S(Y)$. Therefore the Künneth formula yields the "Künneth formula for homology (there is a similar formula for cohomology):"

$$H_n(X \times Y) \cong \left\{ \bigoplus_{p=0}^{n} H_p(X) \otimes H_{n-p}(Y) \right\} \oplus \left\{ \bigoplus_{p=1}^{n} \mathrm{Tor}_1^{\mathbf{Z}}(H_{p-1}(X), H_{n-p}(Y)) \right\}.$$

We now turn to the analogue of the Künneth formula for Hom in place of $\otimes$.

Universal Coefficient Theorem for Cohomology 3.6.5 *Let P be a chain complex of projective R-modules such that each $d(P_n)$ is also projective. Then for every n and every R-module M, there is a (noncanonically) split exact sequence*

$$0 \to \mathrm{Ext}_R^1(H_{n-1}(P), M) \to H^n(\mathrm{Hom}_R(P, M)) \to \mathrm{Hom}_R(H_n(P), M) \to 0.$$

Proof. Since $d(P_n)$ is projective, there is a (noncanonical) isomorphism $P_n \cong Z_n \oplus d(P_n)$ for each n. Therefore each sequence

$$0 \to \mathrm{Hom}(d(P_n), M) \to \mathrm{Hom}(P_n, M) \to \mathrm{Hom}(Z_n, M) \to 0$$

is exact. We may now copy the proof of the Künneth formula 3.6.1 for $\otimes$, using $\mathrm{Hom}(-, M)$ instead of $\otimes M$, to see that the sequence is indeed exact. We may copy the proof of the Universal Coefficient Theorem 3.6.2 for $\otimes$ in the same way to see that the sequence is split. □

Application 3.6.6 (Universal Coefficient theorem in topology) The cohomology of a topological space X with "coefficients" in M is defined to be

$$H^*(X;M) = H^*(\mathrm{Hom}(S(X), M)).$$

In this case, the Universal Coefficient theorem becomes

$$H^n(X;M) \cong \mathrm{Hom}(H_n(X), M) \oplus \mathrm{Ext}^1_{\mathbf{Z}}(H_{n-1}(X), M).$$

Example 3.6.7 If X is path-connected, then $H_0(X) = \mathbf{Z}$ and $H^1(X;\mathbf{Z}) \cong \mathrm{Hom}(H_1(X), \mathbf{Z})$ which is a torsionfree abelian group.

Exercise 3.6.1 Let P be a chain complex and Q a cochain complex of R-modules. As in 2.7.4, form the Hom double cochain complex $\mathrm{Hom}(P, Q) = \{\mathrm{Hom}_R(P_p, Q^q)\}$, and then write $H^* \mathrm{Hom}(P, Q)$ for the cohomology of $\mathrm{Tot}(\mathrm{Hom}(P, Q))$. Show that if each P_n and $d(P_n)$ is projective, there is an exact sequence

$$0 \to \prod_{p+q=n-1} \mathrm{Ext}^1_R(H_p(P), H^q(Q)) \to H^n \mathrm{Hom}(P, Q) \to \prod_{p+q=n} \mathrm{Hom}_R(H_p(P), H^q(Q)) \to 0.$$

Exercise 3.6.2 A ring R is called *right hereditary* if every submodule of every (right) free module is a projective module. (See 4.2.10 and exercise 4.2.6 below.) Any principal ideal domain (for example, $R = \mathbf{Z}$) is hereditary, as is any commutative Dedekind domain. Show that the universal coefficient theorem of this section remain valid if $\mathbf{Z}$ is replaced by an arbitrary right hereditary ring R.

4.1 Dimensions

Definitions 4.1.1 Let A be a right R-module.

1. The *projective dimension* $pd(A)$ is the minimum integer n (if it exists) such that there is a resolution of A by projective modules
$$0 \to P_n \to \cdots \to P_1 \to P_0 \to A \to 0.$$
2. The *injective dimension* $id(A)$ is the minimum integer n (if it exists) such that there is a resolution of A by injective modules
$$0 \to A \to E^0 \to E^1 \to \cdots \to E^n \to 0.$$
3. The *flat dimension* $fd(A)$ is the minimum integer n (if it exists) such that there is a resolution of A by flat modules
$$0 \to F_n \to \cdots \to F_1 \to F_0 \to A \to 0.$$

If no finite resolution exists, we set $pd(A)$, $id(A)$, or $fd(A)$ equal to ∞.

We are going to prove the following theorems in this section, which allow us to define the global and Tor dimensions of a ring R.

Global Dimension Theorem 4.1.2 *The following numbers are the same for any ring R:*

1. $\sup\{id(B) \mid B \in \textbf{mod}\text{-}R\}$
2. $\sup\{pd(A) \mid A \in \textbf{mod}\text{-}R\}$
3. $\sup\left\{pd\left(R/I\right) \mid I \textit{ is a right ideal of } R\right\}$
4. $\sup\{d \mid \operatorname{Ext}^d_R(A,B) \neq 0 \textit{ for some right modules } A, B\}$

This common number (possibly ∞) is called the (right) global dimension *of R, $r.gl.\dim(R)$. Bourbaki [BX] calls it the* homological dimension *of R.*

Remark One may define the left global dimension $\ell.gl.\dim(R)$ similarly. If R is commutative, we clearly have $\ell.gl.\dim(R) = r.gl.\dim(R)$. Equality also holds if R is left *and* right noetherian. Osofsky [Osof] proved that if every one-sided ideal can be generated by at most $\aleph_n$ elements, then $|\ell.gl.\dim(R) - r.gl.\dim(R)| \leq n+1$. The continuum hypothesis of set theory lurks at the fringe of this subject whenever we encounter nonconstructible ideals over uncountable rings.

Tor-dimension Theorem 4.1.3 *The following numbers are the same for any ring R:*

1. $\sup\{fd(A) \mid A \textit{ is a right } R\textit{-module}\}$
2. $\sup\left\{fd\left(R/J\right) \mid J \textit{ is a right ideal of } R\right\}$
3. $\sup\{fd(B) \mid B \textit{ is a left } R\textit{-module}\}$
4. $\sup\left\{fd\left(R/I\right) \mid I \textit{ is a left ideal of } R\right\}$
5. $\sup\{d \mid \operatorname{Tor}^R_d(A,B) \neq 0 \textit{ for some } R\textit{-modules } A, B\}$

This common number (possibly ∞) is called the Tor-dimension *of R. Due to the influence of [CE], the less descriptive name* weak dimension *of R is often used.*

Example 4.1.4 Obviously every field has both global and Tor-dimension zero. The Tor and Ext calculations for abelian groups show that $R = \mathbf{Z}$ has global dimension 1 and Tor-dimension 1. The calculations for $R = {}^{\mathbf{Z}}\!/_{m}$ show that if some $p^2 \mid m$ (so R isn't a product of fields), then ${}^{\mathbf{Z}}\!/_{m}$ has global dimension ∞ and Tor-dimension ∞.

As projective modules are flat, $fd(A) \leq pd(A)$ for every R-module A. We need not have equality: over $\mathbf{Z}$, $fd(\mathbf{Q}) = 0$ by $pd(\mathbf{Q}) = 1$. Taking the supremum over all A shows that Tor-dim$(R) \neq r.gl.\dim(R)$. These examples are perforce non-noetherian, as we now prove, assuming the global and Tor-dimension theorem.

Proposition 4.1.5 *If R is right noetherian, then*

1. $fd(A) = pd(A)$ *for every finitely generated R-module A.*
2. Tor-dim$(R) = r.gl.\dim(R)$.

Proof. Since we can compute Tor-dim(R) and $r.gl.\dim(R)$ using the modules ${}^{R}\!/_{I}$, it suffices to prove (1). Since $fd(A) \leq pd(A)$, it suffices to suppose that $fd(A) = n < \infty$ and prove that $pd(A) \leq n$. As R is noetherian, there is a resolution

$$0 \to M \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A \to 0$$

in which the P_i are finitely generated free modules and M is finitely presented. The fd lemma 4.1.10 below implies that the syzygy M is a flat R-module, so M must also be projective (3.2.7). This proves that $pd(A) \leq n$, as required. □

Exercise 4.1.1 Use the Tor-dimension theorem to prove that if R is both left and right noetherian, then $r.gl.\dim(R) = \ell.gl.\dim(R)$.

Let R be left and right noetherian. Since R is right noetherian, by Proposition 4.1.5, $r.gl.\dim(R) = $ Tor-dim(R). By the Tor-dimension Theorem 4.1.3, Tor-dim$(R) = \sup\left\{fd\left({}^{R}\!/_{I}\right) \mid I \text{ is a left ideal of } R\right\}$.

As R is left noetherian, I is finitely generated. Since I is a left ideal, I is a left R-module, and since I is finitely generated, I is a noetherian left module. Given the left R-module R and left submodule I, R is left noetherian if and only if I and ${}^{R}\!/_{I}$ are left noetherian, so since R is hypothesized left noetherian and I is left noetherian, ${}^{R}\!/_{I}$ is a left noetherian module. Thus all submodules of ${}^{R}\!/_{I}$, in particular ${}^{R}\!/_{I}$ itself, are finitely generated.

By Proposition 4.1.5, $fd(B) = pd(B)$ for every finitely generated R-module B, so certainly for $B = {}^{R}\!/_{I}$ by above. Thus $\sup\left\{fd\left({}^{R}\!/_{I}\right) \mid I \text{ is a left ideal of } R\right\} = \sup\left\{pd\left({}^{R}\!/_{I}\right) \mid I \text{ is a left ideal of } R\right\}$. By the left version of the Global Dimension Theorem 4.1.2, $\sup\left\{pd\left({}^{R}\!/_{I}\right) \mid I \text{ is a left ideal of } R\right\} = \ell.gl.\dim(R)$. Hence,

$$\begin{aligned} r.gl.\dim(R) = \text{Tor-dim}(R) &= \sup\left\{fd\left({}^{R}\!/_{I}\right) \mid I \text{ is a left ideal of } R\right\} \\ &= \sup\left\{pd\left({}^{R}\!/_{I}\right) \mid I \text{ is a left ideal of } R\right\} = \ell.gl.\dim(R). \end{aligned}$$

The pattern of proof for both theorems will be the same, so we begin with the characterization of projective dimension.

pd Lemma 4.1.6 *The following are equivalent for a right R-module A:*

1. *$pd(A) \leq d$.*
2. *$\mathrm{Ext}_R^n(A,B) = 0$ for all $n > d$ and all R-modules B.*
3. *$\mathrm{Ext}_R^{d+1}(A,B) = 0$ for all R-modules B.*
4. *If $0 \to M_d \to P_{d-1} \to P_{d-2} \to \cdots \to P_1 \to P_0 \to A \to 0$ is* any *resolution with the P's projective, then the syzygy M_d is also projective.*

Proof. Since $\mathrm{Ext}^*(A,B)$ may be computed using a projective resolution of A, it is clear that (4) $\implies$ (1) $\implies$ (2) $\implies$ (3). If we are given a resolution of A as in (4), then $\mathrm{Ext}^{d+1}(A,B) \cong \mathrm{Ext}^1(M_d,B)$ by dimension shifting. Now M_d is projective iff $\mathrm{Ext}^1(M_d,B) = 0$ for all B (exercise 2.5.2), so (3) implies (4). $\square$

Example 4.1.7 In 3.1.6 we produced an infinite projective resolution of $A = \mathbf{Z}/p$ over the ring $R = \mathbf{Z}/p^2$. Each syzygy was $\mathbf{Z}/p$, which is not a projective $\mathbf{Z}/p^2$-module. Therefore by (4) we see that $\mathbf{Z}/p$ has $pd = \infty$ over $R = \mathbf{Z}/p^2$. On the other hand, $\mathbf{Z}/p$ has $pd = 0$ over $R = \mathbf{Z}/p$ and $pd = 1$ over $R = \mathbf{Z}$.

The following two lemmas have the same proof as the preceding lemma.

id Lemma 4.1.8 *The following are equivalent for a right R-module B:*

1. *$id(B) \leq d$.*
2. *$\mathrm{Ext}_R^n(A,B) = 0$ for all $n > d$ and all R-modules A.*
3. *$\mathrm{Ext}_R^{d+1}(A,B) = 0$ for all R-modules A.*
4. *If $0 \to B \to E^0 \to \cdots \to E^{d-1} \to M^d \to 0$ is a resolution with the E^i injective, then M^d is also injective.*

Example 4.1.9 In 3.1.6 we gave an infinite injective resolution of $B = \mathbf{Z}/p$ over $R = \mathbf{Z}/p^2$ and showed that $\mathrm{Ext}_R^n\left(\mathbf{Z}/p, \mathbf{Z}/p\right) \cong \mathbf{Z}/p$ for all n. Therefore $\mathbf{Z}/p$ has $id = \infty$ over $R = \mathbf{Z}/p^2$. On the other hand, it has $\mathrm{id} = 0$ over $R = \mathbf{Z}/p$ and $id = 1$ over $\mathbf{Z}$.

fd Lemma 4.1.10 *The following are equivalent for a right R-module A:*

1. *$fd(A) \leq d$.*
2. *$\mathrm{Tor}_n^R(A,B) = 0$ for all $n > d$ and all left R-modules B.*
3. *$\mathrm{Tor}_{d+1}^R(A,B) = 0$ for all left R-modules B.*
4. *If $0 \to M_d \to F_{d-1} \to F_{d-2} \to \cdots \to F_0 \to A \to 0$ is a resolution with the F_i all flat, then M_d is also a flat R-module.*

Lemma 4.1.11 *A left R-module B is injective iff* $\operatorname{Ext}^1\left({}^{R}\!/_{I}, B\right) = 0$ *for all left ideal I.*

Proof. Applying $\operatorname{Hom}(-, B)$ to $0 \to I \to R \to {}^{R}\!/_{I} \to 0$, we see that

$$\operatorname{Hom}(R, B) \to \operatorname{Hom}(I, B) \to \operatorname{Ext}^1\left({}^{R}\!/_{I}, B\right) \to 0$$

is exact. By Baer's criterion 2.3.1, B is injective iff the first map is surjective, that is, iff $\operatorname{Ext}^1\left({}^{R}\!/_{I}, B\right) = 0$. □

Proof of Global Dimension Theorem. The lemmas characterizing $pd(A)$ and $id(A)$ show that $\sup(2) = \sup(4) = \sup(1)$. As $\sup(2) \geq \sup(3)$, we may assume that $d = \sup\left\{pd\left({}^{R}\!/_{I}\right)\right\}$ is finite and that $id(B) > d$ for some R-module B. For this B, choose a resolution

$$0 \to B \to E^0 \to E^1 \to \cdots \to E^{d-1} \to M \to 0$$

with the E's injective. But then for all ideal I we have

$$0 = \operatorname{Ext}_R^{d+1}\left({}^{R}\!/_{I}, B\right) \cong \operatorname{Ext}_R^1\left({}^{R}\!/_{I}, M\right).$$

By the preceding lemma 4.1.11, M is injective, a contradiction to $id(B) > d$. □

Proof of Tor-dimension theorem. The lemma 4.1.10 characterizing $fd(A)$ over R shows that $\sup(5) = \sup(1) \geq \sup(2)$. The same lemma over R^{op} shows that $\sup(5) = \sup(3) \geq \sup(4)$. We may assume that $\sup(2) \leq \sup(4)$, that is, that $d = \sup\left\{fd\left({}^{R}\!/_{J}\right) \mid J \text{ is a right ideal}\right\}$ is at most the supremum over left ideals. We are done unless d is finite and $fd(B) > d$ for some left R-module B. For this B, choose a resolution $0 \to M \to F_{d-1} \to \cdots \to F_0 \to B \to 0$ with the F's flat. But then for all ideals J we have

$$0 = \operatorname{Tor}_{d+1}^R\left({}^{R}\!/_{J}, B\right) \cong \operatorname{Tor}_1^R\left({}^{R}\!/_{J}, M\right).$$

We saw in 3.2.4 that this implies that M is flat, contradicting $fd(B) > d$. □

Exercise 4.1.2 If $0 \to A \to B \to C \to 0$ is an exact sequence, show that

1. $pd(B) \leq \max\{pd(A), pd(C)\}$ with equality except when $pd(C) = pd(A) + 1$.
2. $id(B) \leq \max\{id(A), id(C)\}$ with equality except when $id(A) = id(C) + 1$.
3. $fd(B) \leq \max\{fd(A), fd(C)\}$ with equality except when $fd(C) = fd(A) + 1$.

It is a more careful phrasing of the exercise to say

1. "$pd(B) = \max\{pd(A), pd(C)\}$ or $pd(C) = pd(A) + 1$,"
2. "$id(B) = \max\{id(A), id(C)\}$ or $id(A) = id(C) + 1$," and
3. "$fd(B) = \max\{fd(A), fd(C)\}$ or $fd(C) = fd(A) + 1$."

All three proofs will proceed as follows: (1) show the dimension of B is less than or equal to the max always, and (2) assume that the dimension of C is not one more than the dimension of A, and show that implies the dimension of B is equal to the max.

1. Note that if $\max\{pd(A), pd(C)\} = \infty$, then the inequality is vacuously true (though when we show equality when $pd(C) \neq pd(A) + 1$, we will need to address the infinite case).

   First, we show that $pd(B) \leq \max\{pd(A), pd(C)\}$ always. Suppose that $\max\{pd(A), pd(C)\} = d < \infty$, so $pd(A) \leq d$ and $pd(C) \leq d$. Given the short exact sequence $0 \to A \to B \to C \to 0$, for any R-module D, we have the long exact sequence

$$\begin{tikzcd} & & \ddots \arrow[dll, "\delta"] \\ \operatorname{Ext}_R^{d+1}(C,D) \arrow[r] & \operatorname{Ext}_R^{d+1}(B,D) \arrow[r] & \operatorname{Ext}_R^{d+1}(A,D) \arrow[dll, "\delta"] \\ \ddots & & \end{tikzcd}$$

   By the pd Lemma 4.1.6, since $pd(A), pd(C) \leq d$, $\operatorname{Ext}^n(C, D) = \operatorname{Ext}^n(A, D) = 0$ for all $n > d$ and all R-modules D. So for $n = d + 1$, we thus have

$$\begin{tikzcd} & & \ddots \arrow[dll, "\delta"] \\ 0 \arrow[r] & \operatorname{Ext}_R^{d+1}(B,D) \arrow[r] & 0 \arrow[dll, "\delta"] \\ \ddots, & & \end{tikzcd}$$

   and thus $\operatorname{Ext}^{d+1}(B, D) = 0$ for any D. By the pd Lemma 4.1.6, since $\operatorname{Ext}^{d+1}(B, D) = 0$ for all D, $pd(B) \leq d = \max\{pd(A), pd(C)\}$, as desired.

   We now show equality when $pd(C) \neq pd(A) + 1$. There are four cases where $pd(C) \neq pd(A) + 1$. For the first two cases, we assume the inequality and that all projective dimensions are finite, and show that $pd(B) \geq \max\{pd(A), pd(C)\}$; since $pd(B) \leq \max\{pd(A), pd(C)\}$ by above, this will do it. For the second two cases, we assume the inequality but that one of $pd(A)$ or $pd(C)$ is infinite, so it is enough to show that $pd(B) = \infty = \max\{pd(A), pd(C)\}$. We proceed.

   (a) Suppose that $pd(A) + 1 < pd(C) < \infty$. By the pd Lemma 4.1.6, there exists an R-module D such that $\operatorname{Ext}^{pd(C)}(C, D) \neq 0$ (for if not, then $pd(C) \leq pd(C) - 1$, a contradiction). From the short exact sequence $0 \to A \to B \to C \to 0$, we have the long exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r] & \operatorname{Ext}_R^{pd(C)-1}(A,D) \arrow[dll, "\delta"] \\
\operatorname{Ext}_R^{pd(C)}(C,D) \arrow[r] & \operatorname{Ext}_R^{pd(C)}(B,D) \arrow[r] & \operatorname{Ext}_R^{pd(C)}(A,D) \arrow[dll, "\delta"] \\
\ddots & &
\end{tikzcd}$$

Since $pd(A) + 1 < pd(C)$, we have $pd(A) < pd(C) - 1$, so by the pd Lemma 4.1.6, $\operatorname{Ext}^{pd(C)-1}(A, D) = \operatorname{Ext}^{pd(C)}(A, D) = 0$. Hence by the diagram above, $\operatorname{Ext}^{pd(C)}(B, D) \cong \operatorname{Ext}^{pd(C)}(C, D) \neq 0$, so $pd(B) \geq pd(C) = \max\{pd(A), pd(C)\}$.

(b) Now suppose that $pd(C) < pd(A) + 1 < \infty$. Again by the pd Lemma 4.1.6, there exists an R-module D such that $\operatorname{Ext}^{pd(A)}(A, D) \neq 0$. From the short exact sequence $0 \to A \to B \to C \to 0$, we have the long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & \operatorname{Ext}_R^{pd(A)}(B,D) \arrow[r] & \operatorname{Ext}_R^{pd(A)}(A,D) \arrow[dll, "\delta"] \\
\operatorname{Ext}_R^{pd(A)+1}(C,D) \arrow[r] & \cdots . &
\end{tikzcd}$$

Since $pd(C) < pd(A) + 1$, by the pd Lemma 4.1.6, $\operatorname{Ext}^{pd(A)+1}(C, D) = 0$. Hence by the diagram above, $\operatorname{Ext}^{pd(A)}(B, D) \neq 0$, for if it were zero, then $\operatorname{Ext}^{pd(A)}(A, D) = 0$, a contradiction. Therefore $pd(B) \geq pd(A) = \max\{pd(A), pd(C)\}$.

(c) Now suppose that $pd(C) < pd(A) + 1 = \infty$ (so $pd(A) = \infty$); we need to show that $pd(B) = \infty$. In this case, from the short exact sequence $0 \to A \to B \to C \to 0$, we have the long exact sequence

$$\begin{tikzcd}
 & & \ddots \arrow[dll, "\delta"] \\
\operatorname{Ext}_R^{n}(C,D) \arrow[r] & \operatorname{Ext}_R^{n}(B,D) \arrow[r] & \operatorname{Ext}_R^{n}(A,D) \arrow[dll, "\delta"] \\
\operatorname{Ext}_R^{n+1}(C,D) \arrow[r] & \cdots . &
\end{tikzcd}$$

By the pd Lemma 4.1.6, $\operatorname{Ext}^n(C, D) = 0$ for all $n > pd(C)$. Hence by the diagram above, for all $n > pd(C)$, $\operatorname{Ext}^n(B,D) \cong \operatorname{Ext}^n(A,D)$. Since $pd(A) = \infty$, for all n, there exists $D = D(n)$ depending on n such that $\operatorname{Ext}^n(A, D(n)) \neq 0$. Thus for all $n > pd(C)$, $\operatorname{Ext}^n(B, D(n)) \neq 0$, so by the pd Lemma 4.1.6, $pd(B) = \infty$, as desired.

(d) Now suppose that $pd(A)+1 < pd(C) = \infty$; we need to show that $pd(B) = \infty$. In this case, from the short exact sequence $0 \to A \to B \to C \to 0$, we have the long exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r] & \operatorname{Ext}^n_R(A,D) \arrow[dll, "\delta"] \\
\operatorname{Ext}^{n+1}_R(C,D) \arrow[r] & \operatorname{Ext}^{n+1}_R(B,D) \arrow[r] & \operatorname{Ext}^{n+1}_R(A,D) \arrow[dll, "\delta"] \\
\ddots & &
\end{tikzcd}$$

By the pd Lemma 4.1.6, $\operatorname{Ext}^n(A,D) = 0$ for all $n > pd(A)$. Hence by the diagram above, for all $n > pd(A)$, $\operatorname{Ext}^n(C,D) \cong \operatorname{Ext}^n(B,D)$. Since $pd(C) = \infty$, for all n, there exists $D = D(n)$ depending on n such that $\operatorname{Ext}^n(C, D(n)) \neq 0$. Thus for all $n > pd(A)$, $\operatorname{Ext}^n(B, D(n)) \neq 0$, so by the pd Lemma 4.1.6, $pd(B) = \infty$, as desired.

Therefore, we have equality when $pd(C) \neq pd(A) + 1$.

As an aside, note that the structure of parts 2 and 3 is identical to the structure of part 1, simply replacing the use of the long exact sequence derived from $\operatorname{Ext}(-,D)$ in 1 with $\operatorname{Ext}(D,-)$ in 2 and $\operatorname{Tor}(-,D)$ in 3. We proceed.

2. Note that if $\max\{id(A), id(C)\} = \infty$, then the inequality is vacuously true (though when we show equality when $id(A) \neq id(C) + 1$, we will need to address the infinite case).

   First, we show that $id(B) \leq \max\{id(A), id(C)\}$ always. Suppose that $\max\{id(A), id(C)\} = d < \infty$, so $id(A) \leq d$ and $id(C) \leq d$. Given the short exact sequence $0 \to A \to B \to C \to 0$, for any R-module D, we have the long exact sequence

$$\begin{tikzcd}
& & \ddots \arrow[dll, "\delta" description] \\
\mathrm{Ext}_R^{d+1}(D,A) \arrow[r] & \mathrm{Ext}_R^{d+1}(D,B) \arrow[r] & \mathrm{Ext}_R^{d+1}(D,C) \arrow[dll, "\delta" description] \\
\ddots & &
\end{tikzcd}$$

By the id Lemma 4.1.8, since $id(A), id(C) \leq d$, $\mathrm{Ext}^n(D,A) = \mathrm{Ext}^n(D,C) = 0$ for all $n > d$ and all R-modules D. So for $n = d+1$, we thus have

$$\begin{tikzcd}
& & \ddots \arrow[dll, "\delta" description] \\
0 \arrow[r] & \mathrm{Ext}_R^{d+1}(D,B) \arrow[r] & 0 \arrow[dll, "\delta" description] \\
\ddots, & &
\end{tikzcd}$$

and thus $\mathrm{Ext}^{d+1}(D,B) = 0$ for any D. By the id Lemma 4.1.8, since $\mathrm{Ext}^{d+1}(D,B) = 0$ for all D, $id(B) \leq d = \max\{id(A), id(C)\}$, as desired.

We now show equality when $id(A) \neq id(C) + 1$. There are four cases where $id(A) \neq id(C)+1$. For the first two cases, we assume the inequality and that all injective dimensions are finite, and show that $id(B) \geq \max\{id(A), id(C)\}$; since $id(B) \leq \max\{id(A), id(C)\}$ by above, this will do it. For the second two cases, we assume the inequality but that one of $id(A)$ or $id(C)$ is infinite, so it is enough to show that $id(B) = \infty = \max\{id(A), id(C)\}$. We proceed.

(a) Suppose that $id(C) + 1 < id(A) < \infty$. By the id Lemma 4.1.8, there exists an R-module D such that $\mathrm{Ext}^{id(A)}(D,A) \neq 0$ (for if not, then $id(A) \leq id(A) - 1$, a contradiction). From the short exact sequence $0 \to A \to B \to C \to 0$, we have the long exact sequence

$$
\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{Ext}_R^{id(A)-1}(D,C) \arrow[dll, "\delta"] \\
\mathrm{Ext}_R^{id(A)}(D,A) \arrow[r] & \mathrm{Ext}_R^{id(A)}(D,B) \arrow[r] & \mathrm{Ext}_R^{id(A)}(D,C) \arrow[dll, "\delta"] \\
\ddots & &
\end{tikzcd}
$$

Since $id(C) + 1 < id(A)$, we have $id(C) < id(A) - 1$, so by the id Lemma 4.1.8, $\mathrm{Ext}^{id(A)}(D,C) = \mathrm{Ext}^{id(A)-1}(D,C) = 0$. Hence by the diagram above, $\mathrm{Ext}^{id(A)}(D,B) \cong \mathrm{Ext}^{id(A)}(D,A) \neq 0$, so $id(B) \geq id(A) = \max\{id(A), id(C)\}$.

(b) Now suppose that $id(A) < id(C) + 1 < \infty$. Again by the id Lemma 4.1.8, there exists an R-module D such that $\mathrm{Ext}^{id(C)}(D,C) \neq 0$. From the short exact sequence $0 \to A \to B \to C \to 0$, we have the long exact sequence

$$
\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Ext}_R^{id(C)}(D,B) \arrow[r] & \mathrm{Ext}_R^{id(C)}(D,C) \arrow[dll, "\delta"] \\
\mathrm{Ext}_R^{id(C)+1}(D,A) \arrow[r] & \cdots . &
\end{tikzcd}
$$

Since $id(A) < id(C) + 1$, by the id Lemma 4.1.8, $\mathrm{Ext}^{id(C)+1}(D,A) = 0$. Hence by the diagram above, $\mathrm{Ext}^{id(C)}(D,B) \neq 0$, for if it were zero, then $\mathrm{Ext}^{id(C)}(D,C) = 0$, a contradiction. Therefore $id(B) \geq id(C) = \max\{id(A), id(C)\}$.

(c) Now suppose that $id(A) < id(C) + 1 = \infty$ (so $id(C) = \infty$); we need to show that $id(B) = \infty$. In this case, from the short exact sequence $0 \to A \to B \to C \to 0$, we have the long exact sequence

$$
\begin{tikzcd}
 & & \ddots \arrow[dll, "\delta"] \\
\mathrm{Ext}_R^{n}(D,A) \arrow[r] & \mathrm{Ext}_R^{n}(D,B) \arrow[r] & \mathrm{Ext}_R^{n}(D,C) \arrow[dll, "\delta"] \\
\mathrm{Ext}_R^{n+1}(D,A) \arrow[r] & \cdots . &
\end{tikzcd}
$$

By the id Lemma 4.1.8, $\mathrm{Ext}^n(D,A) = 0$ for all $n > id(A)$. Hence by the diagram above, for all $n > id(A)$, $\mathrm{Ext}^n(D,B) \cong \mathrm{Ext}^n(D,C)$. Since $id(C) = \infty$, for all n, there exists $D = D(n)$ depending on n such that $\mathrm{Ext}^n(D(n),C) \neq 0$. Thus for all $n > id(A)$, $\mathrm{Ext}^n(D(n),B) \neq 0$, so by the id Lemma 4.1.8, $id(B) = \infty$, as desired.

(d) Now suppose that $id(C) + 1 < id(A) = \infty$; we need to show that $id(B) = \infty$. In this case, from the short exact sequence $0 \to A \to B \to C \to 0$, we have the long exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{Ext}^n_R(D,C) \arrow[dll, "\delta"] \\
\mathrm{Ext}^{n+1}_R(D,A) \arrow[r] & \mathrm{Ext}^{n+1}_R(D,B) \arrow[r] & \mathrm{Ext}^{n+1}_R(D,C) \arrow[dll, "\delta"] \\
\ddots & &
\end{tikzcd}$$

By the id Lemma 4.1.8, $\mathrm{Ext}^n(D,C) = 0$ for all $n > id(C)$. Hence by the diagram above, for all $n > id(C)$, $\mathrm{Ext}^n(D,A) \cong \mathrm{Ext}^n(D,B)$. Since $id(A) = \infty$, for all n, there exists $D = D(n)$ depending on n such that $\mathrm{Ext}^n(D(n),A) \neq 0$. Thus for all $n > id(C)$, $\mathrm{Ext}^n(D(n),B) \neq 0$, so by the id Lemma 4.1.8, $id(B) = \infty$, as desired.

Therefore, we have equality when $id(A) \neq id(C) + 1$.

3. Note that if $\max\{fd(A), fd(C)\} = \infty$, then the inequality is vacuously true (though when we show equality when $fd(C) \neq fd(A) + 1$, we will need to address the infinite case).

First, we show that $fd(B) \leq \max\{fd(A), fd(C)\}$ always. Suppose that $\max\{fd(A), fd(C)\} = d < \infty$, so $fd(A) \leq d$ and $fd(C) \leq d$. Given the short exact sequence $0 \to A \to B \to C \to 0$, for any R-module D, we have the long exact sequence

$$\begin{tikzcd}
 & & \ddots \arrow[dll, "\delta"] \\
\mathrm{Tor}^R_{d+1}(A,D) \arrow[r] & \mathrm{Tor}^R_{d+1}(B,D) \arrow[r] & \mathrm{Tor}^R_{d+1}(C,D) \arrow[dll, "\delta"] \\
\ddots & &
\end{tikzcd}$$

By the fd Lemma 4.1.10, since $fd(A), fd(C) \leq d$, $\mathrm{Tor}_n(A,D) = \mathrm{Tor}_n(C,D) = 0$ for all $n > d$ and all R-modules D. So for $n = d+1$, we thus have

$$
\begin{tikzcd}
 & & \ddots \arrow[dll, out=-30, in=150, "\delta"'] \\
0 \arrow[r] & \mathrm{Tor}^R_{d+1}(B,D) \arrow[r] & 0 \arrow[dll, out=-30, in=150, "\delta"'] \\
\ddots,
\end{tikzcd}
$$

and thus $\mathrm{Tor}_{d+1}(B,D) = 0$ for any D. By the fd Lemma 4.1.10, since $\mathrm{Tor}_{d+1}(B,D) = 0$ for all D, $fd(B) \leq d = \max\{fd(A), fd(C)\}$, as desired.

We now show equality when $fd(C) \neq fd(A) + 1$. There are four cases where $fd(C) \neq fd(A)+1$. For the first two cases, we assume the inequality and that all flat dimensions are finite, and show that $fd(B) \geq \max\{fd(A), fd(C)\}$; since $fd(B) \leq \max\{fd(A), fd(C)\}$ by above, this will do it. For the second two cases, we assume the inequality but that one of $fd(A)$ or $fd(C)$ is infinite, so it is enough to show that $fd(B) = \infty = \max\{fd(A), fd(C)\}$. We proceed.

(a) Suppose that $fd(A) + 1 < fd(C) < \infty$. By the fd Lemma 4.1.10, there exists an R-module D such that $\mathrm{Tor}_{fd(C)}(C,D) \neq 0$ (for if not, then $fd(C) \leq fd(C) - 1$, a contradiction). From the short exact sequence $0 \to A \to B \to C \to 0$, we have the long exact sequence

$$
\begin{tikzcd}
 & & \ddots \arrow[dll, out=-30, in=150, "\delta"'] \\
\mathrm{Tor}^R_{fd(C)}(A,D) \arrow[r] & \mathrm{Tor}^R_{fd(C)}(B,D) \arrow[r] & \mathrm{Tor}^R_{fd(C)}(C,D) \arrow[dll, out=-30, in=150, "\delta"'] \\
\mathrm{Tor}^R_{fd(C)-1}(A,D) \arrow[r] & \cdots .
\end{tikzcd}
$$

Since $fd(A) + 1 < fd(C)$, we have $fd(A) < fd(C) - 1$, so by the fd Lemma 4.1.10, $\mathrm{Tor}_{fd(C)-1}(A,D) = \mathrm{Tor}_{fd(C)}(A,D) = 0$. Hence by the diagram above, $\mathrm{Tor}_{fd(C)}(B,D) \cong \mathrm{Tor}_{fd(C)}(C,D) \neq 0$, so $fd(B) \geq fd(C) = \max\{fd(A), fd(C)\}$.

(b) Now suppose that $fd(C) < fd(A) + 1 < \infty$. Again by the fd Lemma 4.1.10, there exists an R-module D such that $\mathrm{Tor}_{fd(A)}(A,D) \neq 0$. From the short exact sequence $0 \to A \to B \to C \to 0$, we have the long exact sequence

$$
\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{Tor}^R_{fd(A)+1}(C,D) \arrow[dll, "\delta"] \\
\mathrm{Tor}^R_{fd(A)}(A,D) \arrow[r] & \mathrm{Tor}^R_{fd(A)}(B,D) \arrow[r] & \cdots .
\end{tikzcd}
$$

Since $fd(C) < fd(A) + 1$, by the fd Lemma 4.1.10, $\mathrm{Tor}_{fd(A)+1}(C, D) = 0$. Hence by the diagram above, $\mathrm{Tor}_{fd(A)}(B, D) \neq 0$, for if it were zero, then $\mathrm{Tor}_{fd(A)}(A, D) = 0$, a contradiction. Therefore $fd(B) \geq fd(A) = \max\{fd(A), fd(C)\}$.

(c) Now suppose that $fd(C) < fd(A) + 1 = \infty$ (so $fd(A) = \infty$); we need to show that $fd(B) = \infty$. In this case, from the short exact sequence $0 \to A \to B \to C \to 0$, we have the long exact sequence

$$
\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{Tor}^R_{n+1}(C,D) \arrow[dll, "\delta"] \\
\mathrm{Tor}^R_{n}(A,D) \arrow[r] & \mathrm{Tor}^R_{n}(B,D) \arrow[r] & \mathrm{Tor}^R_{n}(C,D) \arrow[dll, "\delta"] \\
\ddots & &
\end{tikzcd}
$$

By the fd Lemma 4.1.10, $\mathrm{Tor}_n(C, D) = 0$ for all $n > fd(C)$. Hence by the diagram above, for all $n > fd(C)$, $\mathrm{Tor}_n(A, D) \cong \mathrm{Tor}_n(B, D)$. Since $fd(A) = \infty$, for all n, there exists $D = D(n)$ depending on n such that $\mathrm{Tor}_n(A, D(n)) \neq 0$. Thus for all $n > fd(C)$, $\mathrm{Tor}_n(B, D(n)) \neq 0$, so by the fd Lemma 4.1.10, $fd(B) = \infty$, as desired.

(d) Now suppose that $fd(A) + 1 < fd(C) = \infty$; we need to show that $fd(B) = \infty$. In this case, from the short exact sequence $0 \to A \to B \to C \to 0$, we have the long exact sequence

$$
\begin{tikzcd}
 & & \ddots \arrow[dll, "\delta"] \\
\mathrm{Tor}^R_{n+1}(A,D) \arrow[r] & \mathrm{Tor}^R_{n+1}(B,D) \arrow[r] & \mathrm{Tor}^R_{n+1}(C,D) \arrow[dll, "\delta"] \\
\mathrm{Tor}^R_{n}(A,D) \arrow[r] & \cdots . &
\end{tikzcd}
$$

By the fd Lemma 4.1.10, $\mathrm{Tor}_n(A, D) = 0$ for all $n > fd(A)$. Hence by the diagram above, for all $n > fd(A)$, $\mathrm{Tor}_n(B, D) \cong \mathrm{Tor}_n(C, D)$. Since $fd(C) = \infty$, for all n, there exists $D = D(n)$ depending on n such that $\mathrm{Tor}_n(C, D(n)) \neq 0$. Thus for all $n > fd(A)$, $\mathrm{Tor}_n(B, D(n)) \neq 0$, so by the fd Lemma 4.1.10, $fd(B) = \infty$, as desired.

Therefore, we have equality when $fd(C) \neq fd(A) + 1$.

Exercise 4.1.3

1. Given a (possibly infinite) family $\{A_i\}$ of modules, show that

$$pd\left(\bigoplus A_i\right) = \sup\{pd(A_i)\}.$$

2. Conclude that if S is an R-algebra and P is a projective S-module considered as an R-module, then $pd_R(P) \leq pd_R(S)$.
3. Show that if $r.gl.\dim(R) = \infty$, there actually is an R-module A with $pd(A) = \infty$.

1. First, let $n_i = pd(A_i)$, so by definition, there is a resolution of A_i by projective modules

$$0 \to P_{n_i} \to \cdots \to P_1 \to P_0 \to A_i \to 0$$

for every i. Write $P_\bullet^i$ for the projective resolution $P_\bullet^i \to A_i \to 0$; then $\bigoplus_i P_\bullet^i \to \bigoplus_i A_i \to 0$ is a projective resolution of $\bigoplus A_i$. The length of $\bigoplus P_\bullet^i$ is the supremum over i of the lengths of all $P_\bullet^i$, so since $pd\left(\bigoplus A_i\right)$ is the minimal length projective resolution,

$$pd\left(\bigoplus A_i\right) \leq \sup\{pd(A_i)\}.$$

Conversely, we show

$$\sup\{pd(A_i)\} \leq pd\left(\bigoplus A_i\right);$$

this will complete the proof. If $pd\left(\bigoplus A_i\right) = \infty$, then $\sup\{pd(A_i)\} \leq pd\left(\bigoplus A_i\right)$, so the result follows. Let $pd\left(\bigoplus A_i\right) = n < \infty$, so there is resolution of $\bigoplus A_i$ by projective modules

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to \bigoplus A_i \to 0.$$

For arbitrary fixed i, consider $\pi_i : \bigoplus A_i \to A_i$ the canonical projection. The map π_i is a surjection, so we may append it to the projective resolution of $\bigoplus A_i$, and since the composition of surjections is a surjection, obtain a projective resolution for A_i:

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to \bigoplus A_i \to A_i \to 0$$
$$0 \to P_n \to \cdots \to P_1 \to P_0 \to A_i \to 0.$$

Thus $pd(A_i) \leq n$, but i was arbitrary, so for all i, $pd(A_i) \leq n$, and therefore

$$\sup\{pd(A_i)\} \leq n = pd\left(\bigoplus A_i\right).$$

2. By Proposition 2.2.1, since P is a projective S-module, P is a direct summand of a free S-module. That is, for some S-module Q, $P \oplus Q \cong \bigoplus_i S$. If we consider all S-modules as R-modules by restriction of scalars, by part 1.,

$$pd(S) = \sup_i\{pd(S)\} = pd\left(\bigoplus_i S\right) = pd(P \oplus Q) = \max\{pd(P), pd(Q)\} \geq pd(P).$$

3. Since $r.gl.\dim(R) = \sup\{pd(A) \mid A \text{ is an } R\text{-module}\} = \infty$, we may construct a sequence of R-modules A_i such that for each i, $pd(A_i) \geq i$. It follows that $\infty = \sup\{pd(A_i)\} = pd\left(\bigoplus A_i\right)$ by part 1., so $A = \bigoplus A_i$ is a constructed R-module with $pd(A) = \infty$.

4.2 Rings of Small Dimension

Definition 4.2.1 A ring R is called *(right) semisimple* if every right ideal is a direct summand of R or, equivalently, if R is the direct sum of its minimal ideals. Wedderburn's theorem (see [Lang]) clarifies semisimple rings: they are finite products $R = \prod_{i=1}^{r} R_i$ of matrix rings $R_i = M_{n_i}(D_i) = \mathrm{End}_{D_i}(V_i)$ ($n_i = \dim(V_i)$) over division rings D_i. It follows that right semisimple is the same as left semisimple, and that every semisimple ring is (both left and right) noetherian. By Maschke's theorem, the group ring $k[G]$ of a finite group G over a field k is semisimple if $\mathrm{char}(k)$ doesn't divide the order of G.

Theorem 4.2.2 *The following are equivalent for every ring R, where by "R-module" we mean either left R-module or right R-module.*

1. *R is semisimple.*
2. *R has (left and/or right) global dimension 0.*
3. *Every R-module is projective.*
4. *Every R-module is injective.*
5. *R is noetherian, and every R-module is flat.*

6. *R is noetherian and has* Tor*-dimension* 0.

Proof. We showed in the last section that (2) $\iff$ (3) $\iff$ (4) for left R-modules and also for right R-modules. R is semisimple iff every short exact sequence $0 \to I \to R \to {}^{R}\!/_{I} \to 0$ splits, that is, iff $pd\left({}^{R}\!/_{I}\right) = 0$ for every (right and/or left) ideal I. This proves that (1) $\iff$ (2). As (1) and (3) imply (5), and (5) $\iff$ (6) by definition, we only have to show that (5) implies (1). If I is an ideal of R, then (5) implies that ${}^{R}\!/_{I}$ is finitely presented and flat, hence projective by 3.2.7. Since ${}^{R}\!/_{I}$ is projective, $R \to {}^{R}\!/_{I}$ splits, and I is a direct summand of R, that is, (1) holds. □

Definition 4.2.3 A ring R is *quasi-Frobenius* if it is (left and right) noetherian and R is an injective (left and right) R-module. Our interest in quasi-Frobenius rings stems from the following result of Faith and Faith-Walker, which we quote from [Faith].

Theorem 4.2.4 *The following are equivalent for every ring R:*

1. *R is quasi-Frobenius.*
2. *Every projective right R-module is injective.*
3. *Every injective right R-module is projective.*
4. *Every projective left R-module is injective.*
5. *Every injective left R-module is projective.*

Exercise 4.2.1 Show that ${}^{\mathbf{Z}}\!/_{m}$ is a quasi-Frobenius ring for every $m \neq 0$.

Exercise 4.2.2 Show that if R is quasi-Frobenius, then either R is semisimple or R has global dimension ∞. *Hint*: Every finite projective resolution is split.

Definition 4.2.5 A *Frobenius algebra* over a field k is a finite-dimensional algebra R such that $R \cong \mathrm{Hom}_k(R, k)$ as (right) R-modules. Frobenius algebras are quasi-Frobenius; more generally, $\mathrm{Hom}_k(R, k)$ is an injective R-module for any algebra R over any field k, since k is an injective k-module and $\mathrm{Hom}_k(R, -)$ preserves injectives (being right adjoint to the forgetful functor **mod**-$R \to$ **mod**-k). Frobenius algebras were introduced in 1937 by Brauer and Nesbitt in order to generalize group algebras $k[G]$ of a finite group, especially when $\mathrm{char}(k) = p$ divides the order of G so that $k[G]$ is not semisimple.

Proposition 4.2.6 *If G is a finite group, then $k[G]$ is a Frobenius algebra.*

Proof. Set $R = k[G]$ and define $f : R \to k$ by letting $f(r)$ be the coefficient of $g = 1$ in the unique expression $r = \sum_{g \in G} r_g g$ of every element $r \in k[G]$. Let $\alpha : R \to \mathrm{Hom}_k(R, k)$ be the map $\alpha(r) : x \mapsto f(rx)$. Since $\alpha(r) = fr$, α is a right R-module map; we claim that α is an isomorphism. If $\alpha(r) = 0$ for $r = \sum r_g g$, then $r = 0$ as each $r_g = f(rg^{-1}) = \alpha(r)(g^{-1}) = 0$. Hence α is an injection. As R and $\mathrm{Hom}_k(R, k)$ have the same finite dimension over k, α must be an isomorphism. □

Vista 4.2.7 Let R be a commutative noetherian ring. R is called a *Gorenstein ring* if $id(R)$ is finite; in this case $id(R)$ is the Krull dimension of R, defined in 4.4.1. Therefore a quasi-Frobenius ring is just a Gorenstein ring of Krull dimension zero, and in particular a finite product of 0-dimensional local rings. If R is a 0-dimensional local ring with maximal ideal $\mathfrak{m}$, then R is quasi-Frobenius $\iff$ $\operatorname{ann}_R(\mathfrak{m}) = \{r \in R \mid r\mathfrak{m} = 0\} \cong R/\mathfrak{m}$. If in addition R is finite-dimensional over a field then R is quasi-Frobenius $\iff$ R is Frobenius. This recognition criterion is at the heart of current research into the Gorenstein rings that arise in algebraic geometry.

Now we shall characterize rings of Tor-dimension zero. A ring R is called a *von Neumann regular* if for every $a \in R$ there is an $x \in R$ for which $axa = a$. These rings were introduced by J. von Neumann in 1936 in order to study continuous geometries such as the lattices of projections in "von Neumann algebras" of bounded operators on a Hilbert space. For more information about von Neumann regular rings, see [Good].

Remark A commutative ring R is von Neumann regular iff R has no nilpotent elements and has Krull dimension zero. On the other hand, a commutative ring R is semisimple iff it is a finite product of fields.

Exercise 4.2.3 Show that an infinite product of fields is von Neumann regular. This shows that not every von Neumann regular ring is semisimple.

Exercise 4.2.4 If V is a vector space over a field k (or a division ring k), show that $R = \operatorname{End}_k(V)$ is von Neumann regular. Show that R is semisimple iff $\dim_k(V) < \infty$.

Lemma 4.2.8 *If R is von Neumann regular and I is a finitely generated right ideal of R, then there is an* idempotent e *(an element with $e^2 = e$) such that $I = eR$. In particular, I is a projective R-module, because $R \cong I \oplus (1-e)R$.*

Proof. Suppose first that $I = aR$ and that $axa = a$. It follows that $e = ax$ is idempotent and that $I = eR$. By induction on the number of generators of I, we may suppose that $I = aR + bR$ with $a \in I$ idempotent. Since $bR = abR + (1-a)bR$, we have $I = aR + cR$ for $c = (1-a)b$. If $cyc = c$, then $f = cy$ is idempotent and $af = a(1-a)by = 0$. As fa may not vanish, we consider $e = f(1-a)$. Then $e \in I$, $ae = 0 = ea$, and e is idempotent:

$$e^2 = f(1-a)f(1-a) = f(f-af)(1-a) = f^2(1-a) = f(1-a) = e.$$

Moreover, $eR = cR$ because $c = fc = ffc = f(1-a)fc = efc$. Finally, we claim that I equals $J = (a+e)R$. Since $a+e \in I$, we have $J \subseteq I$; the reverse inclusion follows from the observation that $a = (a+e)a \in J$ and $e = (a+e)e \in J$. $\square$

Exercise 4.2.5 Show that the converse holes: If every fin. gen. right ideal I of R is generated by an idempotent (*i.e.*, $R \cong I \oplus R/I$), then R is von Neumann regular.

Theorem 4.2.9 *The following are equivalent for every ring R:*

1. *R is von Neumann regular.*
2. *R has* Tor-*dimension* 0.
3. *Every R-module is flat.*
4. *${}^{R}\!/_{I}$ is projective for every finitely generated ideal I.*

Proof. By definition, (2) $\iff$ (3). If I is a fin. generated ideal, then ${}^{R}\!/_{I}$ is finitely presented. Thus ${}^{R}\!/_{I}$ is flat iff it is projective, hence iff $R \cong I \oplus {}^{R}\!/_{I}$ as a module. Therefore (3) $\implies$ (4) $\iff$ (1). Finally, any ideal I is the union of its finitely generated subideals I_α, and we have ${}^{R}\!/_{I} = \varinjlim \left({}^{R}\!/_{I_\alpha}\right)$. Hence (4) implies that each ${}^{R}\!/_{I}$ is flat, that is, that (2) holds. $\square$

Remark Since the Tor-dimension of a ring is at most the global dimension, noetherian von Neumann regular rings must be semisimple (4.1.5). Von Neumann regular rings that are not semisimple show that we can have Tor-dim$(R) < gl.\dim(R)$. For example, the global dimension of $\prod_{i=1}^{\infty} \mathbf{C}$ is ≥ 2, with equality iff the Continuum Hypothesis holds.

Defintion 4.2.10 A ring R is called (*right*) *hereditary* if every right ideal is projective. A commutative integral domain R is hereditary iff it is a *Dedekind domain* (noetherian, Krull dimension 0 or 1 and every local ring R_m is a discrete valuation ring). Principal ideal domains (*e.g.*, $\mathbf{Z}$ or $k[t]$) are Dedekind, and of course every semisimple ring is hereditary.

Theorem 4.2.11 *A ring R is right hereditary iff $r.gl.\dim(R) \leq 1$.*

Proof. The exact sequences $0 \to I \to R \to {}^{R}\!/_{I} \to 0$ show that R is hereditary iff $r.gl.\dim(R) \leq 1$. $\square$

Exercise 4.2.6 Show that R is right hereditary iff every submodule of every free module is projective. This was used in exercise 3.6.2.

4.3 Change of Rings Theorems

General Change of Rings Theorem 4.3.1 *Let $f : R \to S$ be a ring map, and let A be an S-module. Then as an R-module*

$$pd_R(A) \leq pd_S(A) + pd_R(S).$$

Proof. There is nothing to prove if $pd_S(A) = \infty$ or $pd_R(S) = \infty$, so assume that $pd_S(A) = n$ and $pd_R(S) = d$ are finite. Choose an S-module projective resolution $Q \to A$ of length n. Starting with R-module projective resolutions of A and of each syzygy in Q, the Horseshoe Lemma 2.2.8 gives us R-module projective resolutions $\widetilde{P}_{*q} \to Q_q$ such that $\widetilde{P}_{*q} \to \widetilde{P}_{*,q-2}$ is zero. We saw in section 4.1 that $pd_R(Q_q) \leq d$ for each q. The truncated resolutions $P_{*q} \to Q_q$ of length d ($P_{iq} = 0$ for $i > d$ and $P_{dq} = {}^{\widetilde{P}_{dq}}\!/_{\mathrm{im}(\widetilde{P}_{d+1,q})}$, as in 1.2.7) have the same property. By the sign trick, we have a double complex P_{**} and an augmentation $P_{0*} \to Q_*$.

$$\begin{array}{c|ccccccccccc}
0 & 0 & & 0 & & & & & & 0 & & \\
\downarrow & \downarrow & & \downarrow & & & & & & \downarrow & & \\
Q_n & P_{0n} & \longleftarrow & P_{1n} & \longleftarrow & \cdots & \longleftarrow & \cdots & \longleftarrow & P_{dn} & \longleftarrow & 0 \\
\downarrow & \downarrow & & \downarrow & & & & & & \downarrow & & \\
\cdots & \cdots & & \cdots & & \cdots & & & & \cdots & & \\
\downarrow & \downarrow & & \downarrow & & & & & & \downarrow & & \\
Q_1 & P_{01} & \longleftarrow & P_{11} & \longleftarrow & P_{21} & \longleftarrow & \cdots & \longleftarrow & P_{d1} & \longleftarrow & 0 \\
\downarrow & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
Q_0 & P_{00} & \longleftarrow & P_{10} & \longleftarrow & P_{20} & \longleftarrow & \cdots & \longleftarrow & P_{d0} & \longleftarrow & 0 \\
\downarrow & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & 0 & & 0 & & 0 & & & & 0 & &
\end{array}$$

The argument used in 2.7.2 to balance Tor shoes that $\mathrm{Tot}(P) \to Q$ is a quasi-isomorphism, because the rows of the augmented double complex (add $Q[-1]$ in column -1) are exact. Hence $\mathrm{Tot}(P) \to A$ is an R-module projective resolution of A. But then $pd_R(A)$ is at most the length of $\mathrm{Tot}(P)$, that is, $d+n$. $\square$

Example 4.3.2 If R is a field and $pd_S(A) \neq 0$, we have strict inequality.

Remark The above argument presages the use of spectral sequences in getting more explicit information about $\mathrm{Ext}^*_R(A,B)$. An important case in which we have equality is the case $S = R/xR$ when x is a nonzerodivisor, so $pd_R\left(R/xR\right) = 1$.

First Change of Rings Theorem 4.3.3 *Let x be a central nonzerodivisor in a ring R. If $A \neq 0$ is a R/x-module with $pd_{R/x}(A)$ finite, then*

$$pd_R(A) = 1 + pd_{R/x}(A).$$

Proof. As $xA = 0$, A cannot be a projective R-module, so $pd_R(A) \geq 1$. On the other hand, if A is a projective R/x-module, then evidently $pd_R(A) = pd_R\left(R/x\right) = 1$. If $pd_{R/x}(A) \geq 1$, find an exact sequence

$$0 \to M \to P \to A \to 0$$

with P a projective R/x-module, so that $pd_{R/x}(A) = pd_{R/x}(M)+1$. By induction, $pd_R(M) = 1+pd_{R/x}(M) = pd_{R/x}(A) \geq 1$. Either $pd_R(A)$ equals $pd_R(M)+1$ or $1 = pd_R(P) = \sup\{pd_R(M), pd_R(A)\}$. We shall conclude the proof by eliminating the possibility that $pd_R(A) = 1 = pd_{R/x}(A)$.

Map a free R-module F onto A with kernel K. If $pd_R(A) = 1$, then K is a projective R-module. Tensoring with R/xR yields the sequence of R/x-modules:

$$0 \to \mathrm{Tor}_1^R\left(A, R/x\right) \to K/xK \to F/xF \to A \to 0.$$

If $pd_{R/x}(A) \leq 2$, then $\mathrm{Tor}_1^R\left(A, R/x\right)$ is a projective R/x-module. But

$$\mathrm{Tor}_1^R\left(A, R/x\right) \cong \{a \in A \mid xa = 0\} = A, \text{ so } pd_{R/x}(A) = 0.$$

$\square$

Example 4.3.4 The conclusion of this theorem fails if $pd_{R/x}(A) = \infty$ but $pd_R(A) < \infty$. For example, $pd_{\mathbf{Z}/4}\left(\mathbf{Z}/2\right) = \infty$ but $pd_{\mathbf{Z}}\left(\mathbf{Z}/2\right) = 1$.

Exercise 4.3.1 Let R be the power series ring $k[[x_1, \cdots, x_n]]$ over a field k. R is a noetherian local ring with residue field k. Show that $gl.\dim(R) = pd_R(k) = n$.

Observe that $k[[x_1, ..., x_n]]\diagup_{(x_n)} \cong k[[x_1, ..., x_{n-1}]]$. We show $pd_{k[[x_1,...,x_n]]}(k) = n$ using the First Change of Rings Theorem 4.3.3 and induction. For the base case when $n = 1$,

$$pd_{k[[x_1]]}(k) = 1 + pd_{k[[x_1]]\diagup_{(x_1)}}(k) = 1 + pd_k(k) = 1 + 0 = 1.$$

Now assume the claim holds for $n - 1$ and observe

$$pd_{k[[x_1,...,x_n]]}(k) = 1 + pd_{k[[x_1,...,x_n]]\diagup_{(x_n)}}(k) = 1 + pd_{k[[x_1,...,x_{n-1}]]}(k) = 1 + n - 1 = n,$$

as desired. Now write $R = k[[x_1, ..., x_n]]$; it remains to be seen that we have $gl.\dim(R) = n$ too. We proceed via double inequality.

First, see that by definition in the Global Dimension Theorem 4.1.2, $gl.\dim(R) = \sup\{pd_R(A) \mid A \text{ is an } R\text{-module}\}$. By our work above, $pd_R(k) = n$, so $gl.\dim(R) = \sup\{pd_R(A)\} \geq n$.

For the inequality in the other direction, note that since k is a field, it is noetherian, and thus $R = k[[x_1, ..., x_n]]$ is noetherian too. Hence by Proposition 4.1.5, $gl.\dim(R) =$ Tor-dim(R), and by Tor-dimension Theorem 4.1.3 (which defines Tor-dimension), Tor-dim$(R) = \sup\{fd_R(A) \mid A \text{ is an } R\text{-module}\}$. We need to show that $\sup\{fd_R(A)\} \leq n$; to prove this, we fix an arbitrary A. It is enough to show that $fd_R(A) \leq n$, and thus as A is arbitrary, $gl.\dim(R) = \sup\{fd_R(A)\} \leq n$.

In the case that A is finitely generated, Proposition 4.1.5 implies that $fd_R(A) = pd_R(A)$, so we show $pd_R(A) \leq n$. Observe that since k is finitely generated, by Proposition 4.1.5, $fd_R(k) = pd_R(k)$, which we saw above is n. By the fd Lemma 4.1.10, this implies $\text{Tor}^R_{n+1}(A, k) = 0$. We claim the following Lemma:

Lemma A Let $R = k[[x_1, ..., x_n]]$ for k a field, and let A be a finitely generated R-module. If $\text{Tor}^R_{n+1}(A, k) = 0$, then $pd_R(A) \leq n$.

Proof. By induction on n. For the base case, let $n = 1$, so that $R = k[[x_1]]$. Let $S = k[[x_1]]\diagup_{(x_1)} \cong k$ so that we have the quotient map $f : R \to S$. By the General

Change of Rings Theorem 4.3.1,

$$pd_R(A) \leq pd_S(A) + pd_R(S) = pd_k(A) + pd_{k[[x_1]]}(k).$$

Since k is a field, a finitely generated module A over k is a vector space, hence projective, and $pd_k(A) = 0$. We claim $pd_{k[[x_1]]}(k) \leq 1$, and prove it via constructing a projective resolution of R-modules of k of length 1. Indeed, we have the short exact sequence

$$0 \to (x_1) \to R \to k \to 0;$$

R is free, hence projective, and thus it is enough to show that (x_1) is a projective R-module. Indeed, we show it in a bit more generality, as we will need in the inductive step:

> **Lemma B** If $R = k[[x_1, ..., x_n]]$, then (x_n) is a projective R-module. Consequently, $pd_R(k[[x_1, ..., x_{n-1}]]) \leq 1$.
>
> *Proof.* The sequence
>
> $$0 \to (x_n) \to R \xrightarrow{\varphi} k[[x_1, ..., x_{n-1}]] \to 0$$
>
> splits, since the map $\psi : k[[x_1, ..., x_n]] \to R$ defined by $\psi\,(p(x_1, ..., x_{n-1})) = p(x_1, ..., x_{n-1}, 0)$ is a map such that $\varphi \circ \psi = \mathrm{id}_{k[[x_1,...,x_{n-1}]]}$. Hence $(x_n) \oplus k[[x_1, ..., x_{n-1}]] \cong R$, so (x_n) is projective. □

So by Lemma B, (x_1) is projective, and we see that $pd_R(k) \leq 1$. Thus, the base case is concluded, since

$$pd_R(A) \leq pd_k(A) + pd_R(k) \leq 0 + 1 = 1.$$

For the inductive step, write $R = k[[x_1, ..., x_n]]$ and $S = k[[x_1, ..., x_{n-1}]]$. Assume the inductive hypothesis: that $\mathrm{Tor}_n^S(A, k) = 0$ implies $pd_S(A) \leq n - 1$. Suppose $\mathrm{Tor}_{n+1}^R(A, k) = 0$. By the General Change of Rings Theorem 4.3.1 and by Lemma B,

$$pd_R(A) \leq pd_S(A) + pd_R(S) \leq n - 1 + 1 = n,$$

as we needed to show. $\square$

Hence by Lemma A, $pd_R(A) \leq n$ for an arbitrary finitely generated R-module A.

In the case that A is not finitely generated, it is a theorem due to Auslander that $pd_R(A) \leq n$ for every R-module A if and only if $pd_R(M) \leq n$ for every finitely generated R-module M. Since the finitely generated case is handled above, by Auslander we have $pd_R(A) \leq n$. Always $fd_R(A) \leq pd_R(A)$, so the result follows.

In either case, $fd_R(A) \leq n$, so $gl.\dim(R) \leq n$, as we wished to show. Hence we may finally conclude that $gl.\dim(R) = n$, and the exercise is complete.

Second Change of Rings Theorem 4.3.5 *Let x be a central nonzerodivisor in a ring R. If A is an R-module and x is a nonzerodivisor on A (i.e., $a \neq 0 \implies xa \neq 0$), then*

$$pd_R(A) \geq pd_{R/x}\left(A/xA\right).$$

Proof. If $pd_R(A) = \infty$, there is nothing to prove, so we assume $pd_R(A) = n < \infty$ and proceed by induction on n. If A is a projective R-module, then A/xA is a projective R/x-module, so the result is true if $pd_R(A) = 0$. If $pd_R(A) \neq 0$, map a free R-module F onto A with kernel K. As $pd_R(K) = n - 1$, $pd_{R/x}\left(K/xK\right) \leq n - 1$ by induction. Tensoring with R/x yields the sequence

$$0 \to \operatorname{Tor}_1^R\left(A, R/x\right) \to K/xK \to F/xF \to A/xA \to 0.$$

As x is a nonzerodivisor on A, $\operatorname{Tor}_1\left(A, R/x\right) \cong \{a \in A \mid xa = 0\} = 0$. Hence either A/xA is projective or $pd_{R/x}\left(A/xA\right) = 1 + pd_{R/x}\left(K/xK\right) \leq 1 + (n-1) = pd_R(A)$. $\square$

Exercise 4.3.2 Use the first Change of Rings Theorem 4.3.3 to find another proof when $pd_{R/x}\left(A/xA\right)$ is finite.

We must show that if A is an R-module and $x \in R$ is central in R and not a zero divisor in A or R, then $pd_R(A) \geq pd_{R/x}\left(A/xA\right)$. Let $pd_{R/x}\left(A/xA\right) < \infty$.

Consider the short exact sequence

$$0 \to A \xrightarrow{x} A \to A/xA \to 0.$$

By our careful rephrasing of Exercise 4.1.2, $pd_R(A) = \max\left\{pd_R(A), pd_R\left(A/xA\right)\right\}$ or $pd_R\left(A/xA\right) = pd_R(A) + 1$. So consider two cases:

1. Assume $pd_R\left(A/xA\right) = pd_R(A) + 1$. In this case, the First Change of Rings Theorem

4.3.3 applied to ${}^{A}\!/_{xA}$ implies

$$pd_R(A)+1 = pd_R\left({}^{A}\!/_{xA}\right) = 1 + pd_{R/x}\left({}^{A}\!/_{xA}\right), \text{ so}$$
$$pd_R(A) = pd_{R/x}\left({}^{A}\!/_{xA}\right).$$

2. Assume $pd_R\left({}^{A}\!/_{xA}\right) \neq pd_R(A)+1$, so that we are forced to have $pd_R(A) = \max\left\{pd_R(A), pd_R\left({}^{A}\!/_{xA}\right)\right\}$. This forces $pd_R(A) \geq pd_R\left({}^{A}\!/_{xA}\right)$. Again applying the First Change of Rings Theorem 4.3.3 to ${}^{A}\!/_{xA}$, we see that

$$pd_R(A) \geq pd_R\left({}^{A}\!/_{xA}\right) = 1 + pd_{R/x}\left({}^{A}\!/_{xA}\right), \text{ so}$$
$$pd_R(A) \geq pd_{R/x}\left({}^{A}\!/_{xA}\right).$$

In either case, $pd_R(A) \geq pd_{R/x}\left({}^{A}\!/_{xA}\right)$, as desired.

Now let $R[x]$ be a polynomial ring in one variable over R. If A is an R-module, write $A[x]$ for the $R[x]$-module $R[x] \otimes_R A$.

Corollary 4.3.6 *$pd_{R[x]}(A[x]) = pd_R(A)$ for every R-module A.*

Proof. Writing $T = R[x]$, we note that x is a nonzerodivisor on $A[x] = T \otimes_R A$. Hence $pd_T(A[x]) \geq pd_R(A)$ by the second Change of Rings theorem 4.3.5. On the other hand, if $P \to A$ is an R-module projective resolution, then $T \otimes_R P \to T \otimes_R A$ is a T-module projective resolution (T is flat over R), so $pd_R(A) \geq pd_T(T \otimes A)$. □

Theorem 4.3.7 *If $R[x_1, \cdots, x_n]$ denotes a polynomial ring in n variables, then $gl.\dim(R[x_1, \cdots, x_n]) = n + gl.\dim(R)$.*

Proof. It suffices to treat the case $T = R[x]$. If $gl.\dim(R) = \infty$, then by the above corollary $gl.\dim(T) = \infty$, so we may assume $gl.\dim(R) = d < \infty$. By the first Change of Rings theorem 4.3.3, $gl.\dim(T) \geq 1 + gl.\dim(R)$. Given a T-module M, write $U(M)$ for the underlying R-module and consider the sequence of T-modules

$$0 \to T \otimes_R U(M) \xrightarrow{\beta} T \otimes_R U(M) \xrightarrow{\mu} M \to 0, \tag{$*$}$$

where μ is multiplication and β is defined by the bilinear map $\beta(t \otimes m) = t[x \otimes m - 1 \otimes (xm)]$ ($t \in T$, $m \in M$). We claim that $(*)$ is exact, which yields the inequality $pd_T(M) \leq 1 + pd_T(T \otimes_R U(M)) = 1 + pd_R(U(M)) \leq 1 + d$. The supremum over all M gives the final inequality $gl.\dim(T) \leq 1 + d$.

To finish the proof, we must establish the claim that $(*)$ is exact. We first observe that, since T is a free R-module on basis $\{1, x, x^2, \cdots\}$, we can write every nonzero element f of $T \otimes U(M)$ as a polynomial with coefficients $m_i \in M$:

$$f = x^k \otimes m_k + \cdots + x^2 \otimes m_2 + x \otimes m_1 + 1 \otimes m_0 \ (m_k \neq 0).$$

Since the leading term of $\beta(f)$ is $x^{k+1} \otimes m_k$, we see that β is injective. Clearly $\mu\beta = 0$. Finally, we prove by induction on k (the degree of f) that if $f \in \ker(\mu)$, then $f \in \operatorname{im}(\beta)$. Since $\mu(1 \otimes m) = m$, the case $k = 0$ is trivial (if $\mu(f) = 0$, then $f = 0$). If $k \neq 0$, then $\mu(f) = \mu(g)$ for the polynomial $f - \beta(x^{k-1} \otimes m_k)$ of lower degree. By induction, if $f \in \ker(\mu)$, then $g = \beta(h)$ for some h, and hence $f = \beta(h + x^{k-1} \otimes m_k)$. □

Corollary 4.3.8 (Hilbert's theorem on syzygys) *If k is a field, then the polynomial ring $k[x_1, \cdots, x_n]$ has global dimension n. Thus the $(n-1)^{st}$ syzygy of every module is a projective module.*

We now turn to the third Change of Rings theorem. For simplicity we deal with commutative local rings, that is, commutative rings with a unique maximal ideal. Here is the fundamental tool used to study local rings.

Nakayama's Lemma 4.3.9 *Let R be a commutative local ring with unique maximal ideal $\mathfrak{m}$ and let B be a nonzero finitely generated R-module. Then*

1. *$B \neq \mathfrak{m}B$.*
2. *If $A \subseteq B$ is a submodule such that $B = A + \mathfrak{m}B$, then $A = B$.*

Proof. If we consider ${}^{B}\!/\!_{A}$ then (2) is a special case of (1). Let m be the smallest integer such that B is generated $b_1, \cdots, b_m$; as $B \neq 0$, we have $m \neq 0$. If $B = \mathfrak{m}B$, then there are $r_i \in \mathfrak{m}$ such that $b_m = \sum r_i b_i$. This yields

$$(1 - r_m)b_m = r_1 b_1 + \cdots + r_{m-1}b_{m-1}.$$

Since $1 - r_m \notin \mathfrak{m}$, it is a unit of R. Multiplying by its inverse writes b_m as a linear combination of $\{b_1, \cdots, b_{m-1}\}$, so this set also generates B. This contradicts the choice of m. $\square$

Remark If R is any ring, the set

$$J = \{r \in R \mid (\forall s \in R) 1 - rs \text{ is a unit of } R\}$$

is a 2-sided ideal of R, called the *Jacobson radical* of R (see [BAII, 4.2]). The above proof actually proves the following:

General Version of Nakayama's Lemma 4.3.10 *Let B be a nonzero finitely generated module over R and J the Jacobson radical of R. Then $B \neq JB$.*

Proposition 4.3.11 *A finitely generated projective module P over a commutative local ring R is a free module.*

Proof. Choose $u_1, \cdots, u_n \in P$ whose images form a basis of the k-vector space ${}^{P}\!/\!_{\mathfrak{m}P}$. By Nakayama's lemma the u's generate P, so the map $\varepsilon : R^n \to P$ sending $(r_1, \cdots, r_n)$ to $\sum r_i u_i$ is onto. As P is projective, ε is split, that is, $R^n \cong P \oplus \ker(\varepsilon)$. As $k^n = {}^{R^n}\!/\!_{\mathfrak{m}R^n} \cong {}^{P}\!/\!_{\mathfrak{m}P}$, we have $\ker(\varepsilon) \subseteq \mathfrak{m}R^n$. But then considering P as a submodule of R^n we have $R^n = P + \mathfrak{m}R^n$, so Nakayama's lemma yields $R^n = P$. $\square$

Third Change of Rings Theorem 4.3.12 *Let R be a commutative noetherian local ring with unique maximal ideal $\mathfrak{m}$, and let A be a finitely generated R-module. If $x \in \mathfrak{m}$ is a nonzerodivisor on both A and R, then*

$$pd_R(A) = pd_{R/x}\left({}^{A}\!/\!_{xA}\right).$$

Proof. We know $\geq$ holds by the second Change of Rings theorem 4.3.5, and we shall prove equality by induction on $n = pd_{R/x}\left({}^{A}\!/\!_{xA}\right)$. If $n = 0$, then ${}^{A}\!/\!_{xA}$ is projective, hence a free ${}^{R}\!/\!_{x}$-module because ${}^{R}\!/\!_{x}$ is local.

Lemma 4.3.13 *If ${}^{A}\!/\!_{xA}$ is a free ${}^{R}\!/\!_{x}$-module, A is a free R-module.*

Proof. Pick elements $u_1, \cdots, u_n$ mapping onto a basis of ${}^{A}\!/\!_{xA}$; we claim they form a basis of A. Since $(u_1, \cdots, u_n)R + xA = A$, Nakayama's lemma states that $(u_1, \cdots, u_n)R = A$, that is, the u's span A. To show the u's are linearly independent, suppose $\sum r_i u_i = 0$ for $r_i \in R$. In ${}^{A}\!/\!_{xA}$, the images of the u's are linearly independent, so $r_i \in xR$ for all i. As x is a nonzerodivisor on R and A, we can divide to get $\frac{r_i}{x} \in R$ such that $\sum \left(\frac{r_i}{x}\right) u_i = 0$. Continuing this process, we get a sequence of elements $r_i, \frac{r_i}{x}, \frac{r_i}{x^2}, \cdots$ which generates a strictly ascending chain of ideals of R, unless $r_i = 0$. As R is noetherian, all the r_i must vanish. $\square$

Resuming the proof of the theorem, we establish the inductive step $n \neq 0$. Map a free R-module F onto A with kernel K. As $\mathrm{Tor}_1^R\left(A, R/x\right) = \{a \in A \mid xa = 0\} = 0$, tensoring with R/x yields the exact sequence

$$0 \to K/xK \to F/xF \to A/xA \to 0.$$

As F/xF is free, $pd_{R/x}\left(K/xK\right) = n - 1$ when $n \neq 0$. As R is noetherian, K is finitely generated, so by induction, $pd_R(K) = n - 1$. This implies that $pd_R(A) = n$, finishing the proof of the third Change of Rings theorem. □

Remark The third Change of Rings theorem holds in the generality that R is right noetherian, and $x \in R$ is a central element lying in the Jacobson radical of R. To prove this, reread the above proof, using the generalized version 4.3.10 of Nakayama's lemma.

Corollary 4.3.14 *Let R be a commutative noetherian local ring, and let A be a finitely generated R-module with $pd_R(A) < \infty$. If $x \in \mathfrak{m}$ is a nonzerodivisor on both A and R, then*

$$pd_R\left(A/xA\right) = 1 + pd_R(A).$$

Proof. Combine the first and third Change of Rings theorems. □

Exercise 4.3.3 (Injective Change of Rings Theorems) Let x be a central nonzerodivisor in a ring R and let A be an R-module. Prove the following.

First Theorem. If $A \neq 0$ is an R/xR-module with $id_{R/xR}(A)$ finite, then

$$id_R(A) = 1 + id_{R/xR}(A).$$

Second Theorem. If x is a nonzerodivisor on both R and A, then either A is injective (in which case $A/xA = 0$) or else

$$id_R(A) \geq 1 + id_{R/xR}\left(A/xA\right).$$

Third Theorem. Suppose that R is a commutative noetherian local ring, A is finitely generated, and that $x \in \mathfrak{m}$ is a nonzerodivisor on both R and A. Then

$$id_R(A) = id_R\left(A/xA\right) = 1 + id_{R/xR}\left(A/xA\right).$$

Proof of First Theorem. First note $id_R(A) \neq 0$ because $xA = 0$, so A cannot be an injective R-module.

Like the projective version, we proceed by induction on $n = id_{R/x}(A)$. The base case is $id_{R/x}(A) = 0$; i.e., A is an injective R/x-module. Let M be an arbitrary R-module, and choose a projective resolution $P_\bullet \to M$. Observe

$$\mathrm{Hom}_R(P_\bullet, A) \cong \mathrm{Hom}_R\left(P_\bullet, \mathrm{Hom}_{R/x}\left(R/x, A\right)\right) \cong \mathrm{Hom}_{R/x}\left(P_\bullet \otimes_R R/x, A\right)$$

by Hom-tensor adjunction. Thus

$$\operatorname{Ext}^i_R(M,A) = H^i(\operatorname{Hom}_R(P_\bullet, A)) \cong H^i\left(\operatorname{Hom}_{R/x}\left(P_\bullet \otimes_R {R/x}, A\right)\right).$$

But $H_i\left(P_\bullet \otimes_R {R/x}\right) = \operatorname{Tor}_i^R\left(M, {R/x}\right) = 0$ for $i > 1$ by Example 3.1.7, as x is not a zero divisor. Thus $\operatorname{Ext}^i_R(M,A) = 0$ for $i > 1$, and thus $id_R(A) \leq 1$. Since $id_R(A) \neq 0$, we conclude $id_R(A) = 1 = 1 + id_{R/x}(A)$, as desired.

For the inductive step, assume the theorem holds for modules with injective dimension at most $k-1$. Let $id_{R/x}(A) = k$. Find an exact sequence

$$0 \to A \to I \to C \to 0$$

with I an injective ${R/x}$-module, so $k = id_{R/x}(A) = id_{R/x}(C) + 1$. By the inductive hypothesis, $id_R(C) = id_{R/x}(C) + 1 = id_{R/x}(A) = k$. By Exercise 4.1.2, either $id_R(A) = id_R(C) + 1 = k + 1$ and we are done, or $1 = id_R(I) = \max\{id_R(C), id_R(A)\} = \max\{k, id_R(A)\} \geq k$, so $k = 1$ (else we are in the base case) and $id_R(A) \neq 0$ means $id_R(A) = 1 = id_{R/x}(A)$. We claim this is impossible; that is, if $id_R(A) = 1$, we show that $id_{R/x}(A) = 0$.

To see this, take J to be injective and consider the short exact sequence $0 \to A \to J \to D \to 0$. Since $id_R(A) = 1$, the cokernel D must be an injective R-module. Thus, taking covariant $\operatorname{Hom}\left({R/x}, -\right)$, we get the exact sequence of ${R/x}$-modules

$$0 \to A \to \operatorname{Hom}\left({R/x}, J\right) \to \operatorname{Hom}\left({R/x}, D\right) \to \operatorname{Ext}^1_R\left({R/x}, A\right) \to 0.$$

If $id_{R/x}(A) \leq 2$, then $\operatorname{Ext}^1\left({R/x}, A\right)$ is an injective ${R/x}$-module, yet $\operatorname{Ext}^1\left({R/x}, A\right) \cong A$, so $id_{R/x}(A) = 0$, as desired to complete the proof. $\square$

Proof of Second Theorem. Like the projective theorem, the proof is by induction on $n = id_R(A)$, which we may assume is finite, else there is nothing to show. Restate the inequality as $id_{R/x}\left({A/xA}\right) \leq id_R(A) - 1$.

For the base case, let $n = 1$ ($n = 0$ implies ${A/xA} = 0$ as in the statement of the theorem). Let I be an injective module and consider the short exact sequence $0 \to A \to I \to C \to 0$. Since

$id_R(A) = 1$, $id_R(C) = 0$, and so ${C}/{xC} = 0$. Taking $\text{Hom}\left({R}/{x}, -\right)$, we get

$$0 \to {A}/{xA} \to {I}/{xI} \to 0.$$

Hence ${A}/{xA}$ is injective. Therefore $0 = id_{R/x}\left({A}/{xA}\right) \leq id_R(A) - 1 = 1 - 1 = 0$.

For the inductive step, let the claim be true for modules with injective dimension at most $k-1$, and let $id_R(A) = k$. Again consider I injective and a short exact sequence $0 \to A \to I \to C \to 0$. Since $id_R(A) = k$, $id_R(C) = k - 1$, and so $id_{R/x}\left({C}/{xC}\right) \leq k - 2$ by the inductive hypothesis. Taking $\text{Hom}\left({R}/{x}, -\right)$, we get

$$0 \to {A}/{xA} \to {I}/{xI} \to {C}/{xC} \to \text{Ext}^1_R\left({R}/{x}, A\right) \to 0.$$

As x is not a zerodivisor in A, $\text{Ext}^1\left({R}/{x}, A\right) = 0$. Hence either ${A}/{xA}$ is injective, or $id_{R/x}\left({A}/{xA}\right) = 1 + id_{R/x}\left({C}/{xC}\right) \leq 1 + k - 2 = k - 1 = id_R(A) - 1$. In either case, $id_{R/x}\left({A}/{xA}\right) \leq id_R(A) - 1$, as we needed to show. $\square$

Proof of Third Theorem. We have $id_R(A) \geq 1 + id_{R/x}\left({A}/{xA}\right)$ by the Second Theorem and $1 + id_{R/x}\left({A}/{xA}\right) = id_R\left({A}/{xA}\right)$ by the First. We proceed by induction on $n = id_{R/x}\left({A}/{xA}\right)$ to show the other inequality: that $id_R(A) - 1 \leq id_{R/x}\left({A}/{xA}\right)$. For the base case, $n = 0$ implies ${A}/{xA}$ is an injective ${R}/{x}$-module. We need to show that $id_R(A) = 1$. Take $0 \to A \to I \to C \to 0$ with I injective and apply $\text{Hom}\left({R}/{x}, -\right)$.

$$0 \to {A}/{xA} \to {I}/{xI} \to {C}/{xC} \to 0.$$

Since ${A}/{xA}$ is injective, ${C}/{xC} = 0$, so C is an injective R-module, and hence $id_R(A) = 1$.

For the inductive step, assume the claim holds for all modules with injective dimension at most $k - 1$ and let $id_{R/x}\left({A}/{xA}\right) = k$. Take the short exact sequence $0 \to A \to I \to C \to 0$ with I injective and apply $\text{Hom}\left({R}/{x}, -\right)$ to get

$$0 \to {A}/{xA} \to {I}/{xI} \to {C}/{xC} \to 0.$$

Since ${I}/{xI}$ is injective, $id_{R/x}\left({C}/{xC}\right) = k-1$. Since R is noetherian, C is finitely generated, so by the inductive hypothesis, $id_R(C) = k - 1$, and thus $id_R(A) = k$, as we needed to show. $\square$

4.4 Local Rings

In this section a *local ring* R will mean a commutative noetherian local ring R with a unique maximal ideal $\mathfrak{m}$. The residue field of R will be denoted $k = {}^{R}\!/_{\mathfrak{m}}$.

Definitions 4.4.1 The *Krull dimension* of a ring R, $\dim(R)$, is the length d of the longest chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_d$ of prime ideals in R; $\dim(R) < \infty$ for every local ring R. The *embedding dimension* of a local ring R is the finite number

$$emb.\dim(R) = \dim_k \left({}^{\mathfrak{m}}\!/_{\mathfrak{m}^2} \right).$$

For any local ring we have $\dim(R) \le emb.\dim(R)$; R is called a *regular local ring* if we have equality, that is, if $\dim(R) = \dim_k \left({}^{\mathfrak{m}}\!/_{\mathfrak{m}^2} \right)$. Regular local rings have been long studied in algebraic geometry because the local coordinate rings of smooth algebraic varieties are regular local rings.

Examples 4.4.2 A regular local ring of dimension 0 must be a field. Every 1-dimensional regular local ring is a discrete valuation ring. The power series ring $k[[x_1, \cdots, x_n]]$ over a field k is regular local of dimension n, as is the local ring $k[x_1, \cdots, x_n]_{\mathfrak{m}}$, $\mathfrak{m} = (x_1, \cdots, x_n)$.

Let R be the local ring of a complex algebraic variety X at a point P. The embedding dimension of R is the smallest integer n such that some analytic neighborhood of P in X embeds in $\mathbf{C}^n$. If the variety X is smooth as a manifold, R is a regular local ring and $\dim(R) = \dim(X)$.

More Definitions 4.4.3 If A is a finitely generated R-module, a *regular sequence on* A, or A*-sequence*, is a sequence $(x_1, \cdots, x_n)$ of elements in $\mathfrak{m}$ such that x_1 is a nonzerodivisor on A (*i.e.*, if $a \neq 0$, then $x_1 a \neq 0$) and such that each x_i $(i > 1)$ is a nonzerodivisor on ${}^{A}\!/_{(x_1, \cdots, x_{i-1})A}$. The *grade* of A, $G(A)$, is the length of the longest regular sequence on A. For any local ring R we have $G(R) \le \dim(R)$.

R is called *Cohen-Macaulay* if $G(R) = \dim(R)$. We will see below that regular local rings are Cohen-Macaulay; in fact, any $x_1, \cdots, x_d \in \mathfrak{m}$ mapping to a basis of ${}^{\mathfrak{m}}\!/_{\mathfrak{m}^2}$ will be an R-sequence; by Nakayama's lemma they will also generate $\mathfrak{m}$ as an ideal. For more details, see [KapCR].

Examples 4.4.4 Every 0-dimensional local ring R is Cohen-Macaulay (since $G(R) = 0$), but cannot be a regular local ring unless R is a field. The 1-dimensional local ring ${}^{k[[x,\varepsilon]]}\!/_{(x\varepsilon = \varepsilon^2 = 0)}$ is not Cohen-Macaulay; every element of $\mathfrak{m} = (x, \varepsilon)R$ kills $\varepsilon \in R$. Unless the maximal ideal consists entirely of zerodivisors, a 1-dimensional local ring R is always Cohen-Macaulay; R is regular only when it is a discrete valuation ring. For example, the local ring $k[[x]]$ is a discrete valuation ring, and the subring $k[[x^2, x^3]]$ is Cohen-Macaulay of dimension 1 but is not a regular local ring.

Exercise 4.4.1 If R is a regular local ring and $x_1, \cdots, x_d \in \mathfrak{m}$ map to a basis of ${}^{\mathfrak{m}}\!/_{\mathfrak{m}^2}$, show that each quotient ring ${}^{R}\!/_{(x_1, \cdots, x_i)R}$ is regular local of dimension $d - i$.

As $(R, \mathfrak{m}, k)$ is a regular local ring, $d = \dim_k \left({}^{\mathfrak{m}}\!/_{\mathfrak{m}^2} \right) = \dim(R)$. For any $i \in \{1, ..., d\}$, the ring $S = {}^{R}\!/_{(x_1, ..., x_i)R}$ is local, because it

1. is the quotient of a commutative ring, hence commutative,
2. is the quotient of a noetherian ring, hence noetherian, and
3. has maximal ideal $\mathfrak{n} = {}^{\mathfrak{m}}\!/_{(x_1, ..., x_i)R}$ by the fourth ring isomorphism theorem, which says $\mathfrak{m}$ is an ideal of R if and only if ${}^{\mathfrak{m}}\!/_{(x_1, ..., x_i)R}$ is an ideal of ${}^{R}\!/_{(x_1, ..., x_i)R}$.

The ideal $\mathfrak{n}$ is maximal because the third ring isomorphism theorem implies

$$S/\mathfrak{n} = \left(R/(x_1, ..., x_i)R\right) \Big/ \left(\mathfrak{m}/(x_1, ..., x_i)R\right) \cong R/\mathfrak{m} \cong k.$$

We next must show S is regular; i.e., $\dim(S) = \dim_k \left(\mathfrak{n}/\mathfrak{n}^2\right)$. Since S is local, $\dim(S) \leq emb.\dim(S)$. First observe that

$$\begin{aligned}
\dim_k \left(\mathfrak{n}/\mathfrak{n}^2\right) &= \dim_k \left(\left(\mathfrak{m}/(x_1, ..., x_i)R\right) \Big/ \left(\mathfrak{m}/(x_1, ..., x_i)R\right)^2 \right) \\
&= \dim_k \left(\left(\mathfrak{m}/(x_1, ..., x_i)R\right) \Big/ \left(\mathfrak{m}^2/(x_1, ..., x_i)R\right) \right) \\
&= \dim_k \left(\mathfrak{m}/\left(\mathfrak{m}^2 + (x_1, ..., x_i)R\right) \right) \\
&= \dim_k \left(\left(\mathfrak{m}/\mathfrak{m}^2\right) \Big/ \left((x_1, ..., x_i)R/\mathfrak{m}^2\right) \right) \\
&= \dim_k \left(\mathfrak{m}/\mathfrak{m}^2\right) - \dim_k \left((x_1, ..., x_i)R/\mathfrak{m}^2\right) \\
&= d - i,
\end{aligned}$$

so $\dim(S) \leq emb.\dim(S) = d - i$. If we can show $d - i \leq \dim(S)$, then equality is forced and we are done. We cite the following claim, so that the result for all $i \in \{1, ..., d\}$ will follow by induction:

> **Lemma** [\tag\00KW, The Stacks project]. If $(R, \mathfrak{m}, k)$ is a local ring and $x \in \mathfrak{m}$, then $\dim(R) - 1 \leq \dim \left(R/(x)R\right)$.

The desired result then follows by induction; for the base case, the Lemma above shows it directly, and for the inductive step, $\left(R/(x_1, ..., x_{i-1})R, \mathfrak{m}/(x_1, ..., x_{i-1})R, k\right)$ is a local ring by above and $x_i \in \mathfrak{m}/(x_1, ..., x_{i-1})R$, so $d - i = d - (i-1) - 1 \leq \dim \left(R/(x_1, ..., x_{i-1})R\right) - 1 \leq \dim \left(R/(x_1, ..., x_i)R\right)$.

Proposition 4.4.5 *A regular local ring is an integral domain.*

Proof. We use induction on $\dim(R)$. Pick $x \in \mathfrak{m} \setminus \mathfrak{m}^2$; by the above exercise, R/xR is regular local of dimension $\dim(R) - 1$. Inductively, R/xR is a domain, so xR is a prime ideal. If there is a prime ideal Q properly contained in xR, then $Q \subset x^n R$ for all n (inductively, if $q = rx^n \in Q$, then $r \in Q \subset xR$, so $q \in x^{n+1}R$). In this case $Q \subseteq \cap x^n R = 0$, whence $Q = 0$ and R is a domain. If R were not a domain, this would imply that xR is a minimal prime ideal of R for all $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. Hence $\mathfrak{m}$ would be contained in the union of $\mathfrak{m}^2$ and the finitely many minimal prime ideals $P_1, \cdots, P_t$ of R. This would imply that $\mathfrak{m} \subseteq P_i$ for some i. But then $\dim(R) = 0$, a contradiction. ☐

Corollary 4.4.6 *If R is a regular local ring, then $G(R) = \dim(R)$, and any $x_1, \cdots, x_d \in \mathfrak{m}$ mapping to a basis of $\mathfrak{m}/\mathfrak{m}^2$ is an R-sequence.*

Proof. As $G(R) \leq \dim(R)$, and $x_1 \in R$ is a nonzerodivisor on R, it suffices to prove that $x_2, \cdots, x_d$ form a regular sequence on R/x_1R. This follows by induction on d. ☐

Exercise 4.4.2 Let R be a regular local ring and I an ideal such that R/I is also regular local. Prove that $I = (x_1, \cdots, x_i)R$, where $(x_1, \cdots, x_i)$ form a regular sequence in R.

Let $\mathfrak{n} = \mathfrak{m}/I$ be the maximal ideal of R/I, and let k be the residue field of R and R/I. If we say $\dim(R) = d$, then for some $i \in \{0, ..., d\}$, $\dim\left(R/I\right) = d - i$. Consider the surjection

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{f} \mathfrak{n}/\mathfrak{n}^2 \to 0$$

defined by $x + \mathfrak{m}^2 \overset{f}{\mapsto} [x] + \mathfrak{n}^2$, where $[x]$ is the equivalence class of x in $\mathfrak{n} = \mathfrak{m}/I$. Observe that

$$\begin{aligned}\ker f &= \left\{x + \mathfrak{m}^2 \in \mathfrak{m}/\mathfrak{m}^2 \mid [x] + \mathfrak{n}^2 = [0] + \mathfrak{n}^2\right\} \\ &= \left\{x + \mathfrak{m}^2 \mid x \in \mathfrak{m}^2 \text{ or } x \in I\right\},\end{aligned}$$

To see this, the inclusion $\left\{x + \mathfrak{m}^2 \mid x \in \mathfrak{m}^2 \text{ or } x \in I\right\} \subseteq \ker f$ is because if $x \in \mathfrak{m}^2$, $x + \mathfrak{m}^2 = 0 + \mathfrak{m}^2 \mapsto 0$, and if $x \in I$, $x + \mathfrak{m}^2 \mapsto [x] + \mathfrak{n}^2 = (x + I) + \mathfrak{n}^2 = (0 + I) + \mathfrak{n}^2 = 0$. The inclusion $\ker f \subseteq \left\{x + \mathfrak{m}^2 \mid x \in \mathfrak{m}^2 \text{ or } x \in I\right\}$ is because if $[x] + \mathfrak{n}^2 = [0] + \mathfrak{n}^2 = (0 + I) + \mathfrak{n}^2$ but $x + \mathfrak{m}^2 \neq 0$, i.e., $x \notin \mathfrak{m}^2$, then $x \in I$.

Hence, $\ker f = \left\{x + \mathfrak{m}^2 \mid x \in \mathfrak{m}^2 \text{ or } x \in I\right\} = \left(I + \mathfrak{m}^2\right)/\mathfrak{m}^2$. Therefore, we have the short exact sequence

$$0 \to \left(I + \mathfrak{m}^2\right)/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{f} \mathfrak{n}/\mathfrak{n}^2 \to 0.$$

Consequently, by Rank-Nullity, $\dim_k\left(\mathfrak{m}/\mathfrak{m}^2\right) = \dim_k\left(\left(I + \mathfrak{m}^2\right)/\mathfrak{m}^2\right) + \dim_k\left(\mathfrak{n}/\mathfrak{n}^2\right)$, so since

R and R/I are regular local rings,

$$\dim_k\left((I+\mathfrak{m}^2)/\mathfrak{m}^2\right) = \dim_k\left(\mathfrak{m}/\mathfrak{m}^2\right) - \dim_k\left(\mathfrak{n}/\mathfrak{n}^2\right)$$
$$= \dim(R) - \dim\left(R/I\right)$$
$$= d-(d-i) = i.$$

So there exists a basis $x_1, ..., x_i$ of I; i.e., their images in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent. By Corollary 4.4.6, we may choose additional regular elements $x_{i+1}, ..., x_d \in \mathfrak{m}$ to get a sequence whose images form a basis of all of $\mathfrak{m}/\mathfrak{m}^2$. Thus, by the universal mapping property, the map $\varphi : R/(x_1, ..., x_i)R \to R/I$ is a surjection, so by the first ring isomorphism theorem,

$$R/I \cong \left(R/(x_1, ..., x_i)R\right)/\ker\varphi,$$

and thus $\dim\left(R/I\right) \leq \dim\left(R/(x_1, ..., x_i)R\right)$. Yet $\dim\left(R/(x_1, ..., x_i)R\right) = d-i$ by Exercise 4.4.1, and $\dim\left(R/I\right) = d-i$ by hypothesis. Hence $\ker\varphi = 0$, and thus $R/I \cong R/(x_1, ..., x_i)$, as desired.

Standard Facts 4.4.7 Part of the standard theory of associated prime ideals in commutative noetherian rings implies that if every element of $\mathfrak{m}$ is a zerodivisor on a finitely generated R-module A, then $\mathfrak{m}$ equals $\{r \in R \mid ra = 0\}$ for some nonzero $a \in A$ and therefore $aR \cong R/\mathfrak{m} = k$. Hence if $G(A) = 0$, then $\mathrm{Hom}_R(k, A) \neq 0$.

If $G(A) \neq 0$ and $G(R) \neq 0$, then some element of $\mathfrak{m} \setminus \mathfrak{m}^2$ must also be a nonzerodivisor on both R and A. Again, this follows from the standard theory of associated prime ideals. Another standard fact is that if $x \in \mathfrak{m}$ is a nonzerodivisor on R, then the Krull dimension of R/xR is $\dim(R) - 1$.

Theorem 4.4.8 *If R is a local ring and $A \neq 0$ is a finitely generated R-module, then every maximal A-sequence has the same length, $G(A)$. Moreover, $G(A)$ is characterized as the smallest n such that $\mathrm{Ext}^n_R(k, A) \neq 0$.*

Proof. We saw above that if $G(A) = 0$, then $\mathrm{Hom}_R(k, A) \neq 0$. Conversely, if $\mathrm{Hom}_R(k, A) \neq 0$, then some nonzero $a \in A$ has $aR \cong k$, that is, $ax = 0$ for all $x \in \mathfrak{m}$. In this case $G(A) = 0$ is clear. We now proceed by induction on the length n of a maximal regular A-sequence $x_1, \cdots, x_n$ on A. If $n \geq 1$, $x = x_1$ is a nonzerodivisor on A, so the sequence $0 \to A \xrightarrow{x} A \to A/xA \to 0$ is exact, and $x_2, \cdots, x_n$ is a maximal regular sequence on A/xA. This yields the exact sequence

$$\mathrm{Ext}^{i-1}(k, A) \xrightarrow{x} \mathrm{Ext}^{i-1}(k, A) \to \mathrm{Ext}^{i-1}\left(k, A/xA\right) \to \mathrm{Ext}^i(k, A) \xrightarrow{x} \mathrm{Ext}^i(k, A).$$

Now $xk = 0$, so $\mathrm{Ext}^i(k, A)$ is an R/xR-module. Hence the maps "x" in this sequence are zero. By induction, this proves that $\mathrm{Ext}^i(k, A) = 0$ for $0 \leq i < n$ and that $\mathrm{Ext}^n(k, A) \neq 0$. This finishes the inductive step, proving the theorem. $\square$

Remark The injective dimension $id(A)$ is the largest integer n such that $\mathrm{Ext}^n_R(k,A) \neq 0$. This follows from the next result, which we cite without proof from [KapCR, section 4.5] because the proof involves more ring theory than we want to use.

Theorem 4.4.9 *If R is a local ring and A is a finitely generated R-module, then*

$$id(A) \leq d \iff \mathrm{Ext}^n_R(k,A) = 0 \text{ for all } n > d.$$

Corollary 4.4.10 *If R is a Gorenstein local ring (i.e., $id_R(R) < \infty$), then R is also Cohen-Macaulay. In this case $G(R) = id_R(R) = \dim(R)$ and*

$$\mathrm{Ext}^q_R(k,R) \neq 0 \iff q = \dim(R).$$

Proof. The last two theorems imply that $G(R) \leq id(R)$, and $id(R) = \dim(R)$ by 4.2.7. Now suppose that $G(R) = 0$ but that $id(R) \neq 0$. For each $s \in R$ and $n \geq 0$ we have an exact sequence

$$\mathrm{Ext}^n_R(R,R) \to \mathrm{Ext}^n_R(sR,R) \to \mathrm{Ext}^{n+1}_R\left(R/sR, R\right).$$

For $n = id(R) > 0$, the outside terms vanish, so $\mathrm{Ext}^n_R(sR,R) = 0$ as well. Choosing $s \in R$ so that $sR \cong k$ contradicts the previous theorem so if $G(R) = 0$ then $id(R) = 0$. If $G(R) = d > 0$, choose a nonzerodivisor $x \in \mathfrak{m}$ and set $S = R/xR$. By the third Injective Change of Rings theorem (exercise 4.3.3), $id_S(S) = id_R(R) - 1$, so S is also a Gorenstein ring. Inductively, S is Cohen-Macaulay, and $G(S) = id_S(S) = \dim(S) = \dim(R) - 1$. Hence $id_R(R) = \dim(R)$. If $x_2, \cdots, x_d$ are elements of $\mathfrak{m}$ mapping onto a maximal S-sequence in $\mathfrak{m}S$, then $x_1, x_2, \cdots, x_d$ forms a maximal R-sequence, that is, $G(R) = 1 + G(S) = \dim(R)$. □

Proposition 4.4.11 *If R is a local ring with residue field k, then for every finitely generated R-module A and every integer d*

$$pd(A) \leq d \iff \mathrm{Tor}^R_{d+1}(A,k) = 0.$$

In particular, $pd(A)$ is the largest d such that $\mathrm{Tor}^R_d(A,k) \neq 0$.

Proof. As $fd(A) \leq pd(A)$, the $\Longrightarrow$ direction is clear. We prove the converse by induction on d. Nakayama's lemma 4.3.9 states that the finitely generated R-module A can be generated by $m = \dim_k\left(A/\mathfrak{m}A\right)$ elements. Let $\{u_1, \cdots, u_m\}$ be a minimal set of generators for A, and let K be the kernel of the surjection $\varepsilon : R^m \to A$ defined by $\varepsilon(r_1, \cdots, r_m) = \sum r_i u_i$. The inductive step is clear, since if $d \neq 0$, then

$$\mathrm{Tor}_{d+1}(A,k) = \mathrm{Tor}_d(K,k) \text{ and } pd(A) \leq 1 + pd(K).$$

If $d = 0$, then the assumption that $\mathrm{Tor}_1(A,k) = 0$ gives exactness of

$$\begin{tikzcd}
0 \arrow[r] & K \otimes k \arrow[r] \arrow[d, equal] & R^m \otimes k \arrow[r] \arrow[d, equal] & A \otimes k \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K/\mathfrak{m}K \arrow[r] & k^m \arrow[r, "\varepsilon\otimes k"] & A/\mathfrak{m}A \arrow[r] & 0.
\end{tikzcd}$$

By construction, the map $\varepsilon \otimes k$ is an isomorphism. Hence $K/\mathfrak{m}K = 0$, so the finitely generated R-module K must be zero by Nakayama's lemma. This forces $R^m \cong A$, so $pd(A) = 0$ as asserted. □

Corollary 4.4.12 *If R is a local ring, then $gl.\dim(R) = pd_R\left(R/\mathfrak{m}\right)$.*

Proof. $pd\left(R/\mathfrak{m}\right) \leq gl.\dim(R) = \sup\left\{pd\left(R/I\right)\right\} \leq fd\left(R/\mathfrak{m}\right) \leq pd\left(R/\mathfrak{m}\right)$. □

Corollary 4.4.13 *If R is local and $x \in \mathfrak{m}$ is a nonzerodivisor on R, then either $gl.\dim\left(R/xR\right) = \infty$ or $gl.\dim(R) = 1 + gl.\dim\left(R/xR\right)$.*

Proof. Set $S = R/xR$ and suppose that $gl.\dim(S) = d$ is finite. By the First Change of Rings Theorem, the residue field $k = R/\mathfrak{m} = S/\mathfrak{m}S$ has

$$pd_R(k) = 1 + pd_S(k) = 1 + d.$$

□

Grade 0 Lemma 4.4.14 *If R is local and $G(R) = 0$ (i.e., every element of the maximal ideal $\mathfrak{m}$ is a zerodivisor on R), then for any finitely generated R-module A,*

$$\textit{either } pd(A) = 0 \textit{ or } pd(A) = \infty.$$

Proof. If $0 < pd(A) < \infty$ for some A then an appropriate syzygy M of A is finitely generated and has $pd(M) = 1$. Nakayama's lemma states that M can be generated by $m = \dim_k\left(M/\mathfrak{m}M\right)$ elements. If $u_1, \cdots, u_m$ generate M, there is a projective resolution $0 \to P \to R^m \xrightarrow{\varepsilon} M \to 0$ with $\varepsilon(r_1, ..., r_m) = \sum r_i u_i$; visibly $R^m/\mathfrak{m}R^m \cong k^m \cong M/\mathfrak{m}M$. But then $P \subseteq \mathfrak{m}R^m$, so $sP = 0$, where $s \in R$ is any element such that $\mathfrak{m} = \{r \in R \mid sr = 0\}$. On the other hand, P is projective, hence a free R-module (4.3.11), so $sP = 0$ implies that $s = 0$, a contradiction. □

Theorem 4.4.15 (Auslander-Buchsbaum Equality) *Let R be a local ring, and A a finitely generated R-module. If $pd(A) < \infty$, then $G(R) = G(A) + pd(A)$.*

Proof. If $G(R) = 0$ and $pd(A) < \infty$, then A is projective (hence free) by the Grade 0 lemma 4.4.14. In this case $G(R) = G(A)$, and $pd(A) = 0$. If $G(R) \neq 0$, we shall perform a double induction on $G(R)$ and on $G(A)$.

Suppose first that $G(R) \neq 0$ and $G(A) = 0$. Choose $x \in \mathfrak{m}$ and $0 \neq a \in A$ so that x is a nonzerodivisor on R and $\mathfrak{m}a = 0$. Resolve A:

$$0 \to K \to R^m \xrightarrow{\varepsilon} A \to 0$$

and choose $u \in R^m$ with $\varepsilon(u) = a$. Now $\mathfrak{m}u \subseteq K$ so $xu \in K$ and $\mathfrak{m}(xu) \subseteq xK$, yet $xu \notin xK$ as $u \notin K$ and x is a nonzerodivisor on R^m. Hence $G\left(K/xK\right) = 0$. Since K is a submodule of a free module, x is a nonzerodivisor on K. By the third Change of Rings theorem, and the fact that A is not free (as $G(R) \neq G(A)$),

$$pd_{R/xR}\left(K/xK\right) = pd_R(K) = pd_R(A) - 1.$$

Since $G\left(R/xR\right) = G(R) - 1$, induction gives us the required identity:

$$G(R) = 1 + G\left(R/xR\right) = 1 + G\left(K/xK\right) + pd_{R/xR}\left(K/xK\right) = pd_R(A).$$

Finally, we consider the case $G(R) \neq 0$, $G(A) \neq 0$. We can pick $x \in \mathfrak{m}$, which is a nonzerodivisor on both R and A (see the *Standard Facts* 4.4.7 cited above). Since we may begin a maximal A-sequence with x, $G\left(A/xA\right) = G(A) - 1$. Induction and the corollary 4.3.14 to the third Change of Rings theorem now give us the required identity:

$$\begin{aligned} G(R) &= G\left(A/xA\right) + pd_R\left(A/xA\right) \\ &= (G(A) - 1) + (1 + pd_R(A)) \\ &= G(A) + pd_R(A). \end{aligned}$$

□

Main Theorem 4.4.16 *A local ring R is regular iff $gl.\dim(R) < \infty$. In this case*

$$G(R) = \dim(R) = emb.\dim(R) = gl.\dim(R) = pd_R(k).$$

Proof. First, suppose R is regular. If $\dim(R) = 0$, R is a field, and the result is clear. If $d = \dim(R) > 0$, choose an R-sequence $x_1, \cdots, x_d$ generating $\mathfrak{m}$ and set $S = R/x_1R$. Then $x_2, \cdots, x_d$ is an S-sequence generating the maximal ideal of S, so S is regular of dimension $d-1$. By induction on d, we have

$$gl.\dim(R) = 1 + gl.\dim(S) = 1 + (d-1) = d.$$

If $gl.\dim(R) = 0$, R must be semisimple and local (a field). If $gl.\dim(R) \neq 0, \infty$, then $\mathfrak{m}$ contains a nonzerodivisor x by the Grade 0 lemma 4.4.14; we may even find an $x = x_1$ not in $\mathfrak{m}^2$ (see the *Standard Facts* 4.4.7 cited above). To prove that R is regular, we will prove that $S = R/xR$ is regular; as $\dim(S) = \dim(R) - 1$, this will prove that the maximal ideal $\mathfrak{m}S$ of S is generated by an S-sequence $y_2, \cdots, y_d$. Lift the $y_i \in \mathfrak{m}S$ to elements $x_i \in \mathfrak{m}$ $(i = 2, \cdots, d)$. By definition $x_1, \cdots, x_d$ is an R-sequence generating $\mathfrak{m}$, so this will prove that R is regular.

By the third Change of Rings theorem 4.3.12 with $A = \mathfrak{m}$,

$$pd_S\left(\mathfrak{m}/x\mathfrak{m}\right) = pd_R(\mathfrak{m}) = pd_R(k) - 1 = gl.\dim(R) - 1.$$

Now the image of $\mathfrak{m}/x\mathfrak{m}$ in $S = R/xR$ is $\mathfrak{m}/xR = \mathfrak{m}S$, so we get exact sequences

$$0 \to xR/x\mathfrak{m} \to \mathfrak{m}/x\mathfrak{m} \to \mathfrak{m}S \to 0 \text{ and } 0 \to \mathfrak{m}S \to S \to k \to 0.$$

Moreover, $xR/x\mathfrak{m} \cong \mathrm{Tor}_1^R\left(R/xR, k\right) \cong \{a \in k \mid xa = 0\} = k$, and the image of x in $xR/x\mathfrak{m}$ is nonzero. We claim that $\mathfrak{m}/x\mathfrak{m} \cong \mathfrak{m}S \oplus k$ as S-modules. This will imply that

$$gl.\dim(S) = pd_S(k) \leq pd_S\left(\mathfrak{m}/x\mathfrak{m}\right) = gl.\dim(R) - 1.$$

By induction on global dimension, this will prove that S is regular.

To see the claim, set $r = emb.\dim(R)$ and find elements $x_2, \cdots, x_r$ in $\mathfrak{m}$ such that the image of $\{x_1, \cdots, x_r\}$ in $\mathfrak{m}/\mathfrak{m}^2$ forms a basis. Set $I = (x_2, \cdots, x_r)R + x\mathfrak{m}$ and observe that $I/x\mathfrak{m} \subseteq \mathfrak{m}/x\mathfrak{m}$ maps onto $\mathfrak{m}S$. As the kernel $xR/x\mathfrak{m}$ of $\mathfrak{m}/x\mathfrak{m} \to \mathfrak{m}S$ is isomorphic to k and contains $x \notin I$, it follows that $\left(xR/x\mathfrak{m}\right) \cap \left(I/x\mathfrak{m}\right) = 0$. Hence $I/x\mathfrak{m} \cong \mathfrak{m}S$ and $k \oplus \mathfrak{m}S \cong \mathfrak{m}/x\mathfrak{m}$, as claimed. □

Corollary 4.4.17 *A regular ring is both Gorenstein and Cohen-Macaulay.*

Corollary 4.4.18 *If R is a regular local ring and $\mathfrak{p}$ is any prime ideal of R, then the localization $R_\mathfrak{p}$ is also a regular local ring.*

Proof. We shall show that if S is any multiplicative set in R, then the localization $S^{-1}R$ has finite global dimension. As $R_\mathfrak{p} = S^{-1}R$ for $S = R \setminus \mathfrak{p}$, this will suffice. Considering an $S^{-1}R$-module A as an R-module, there is a projective resolution $P \to A$ of length at most $gl.\dim(R)$. Since $S^{-1}R$ is a flat R-module and $S^{-1}A = A$, $S^{-1}P \to A$ is a projective $S^{-1}R$-module resolution of length at most $gl.\dim(R)$. □

Remark The only non-homological proof of this result, due to Nagata, is very long and hard. This ability of homological algebra to give easy proofs of results outside the scope of homological algebra justifies its importance. Here is another result, quoted without proof from [KapCR], which uses homological algebra (projective resolutions) in the proof but not in the statement.

Theorem 4.4.19 *Every regular local ring is a Unique Factorization Domain.*

4.5 Koszul Complexes

An efficient way to perform calculations is to use Koszul complexes. If $x \in R$ is central, we let $K(x)$ denote the chain complex

$$0 \to R \xrightarrow{x} R \to 0$$

concentrated in degrees 1 and 0. It is convenient to identify the generator of the degree 1 part of $K(x)$ as the element e_x, so that $d(e_x) = x$. If $\boldsymbol{x} = (x_1, \cdots, x_n)$ is a finite sequence of central elements in R, we define the *Koszul complex* $K(\boldsymbol{x})$ to be the total tensor product complex (see 2.7.1):

$$K(x_1) \otimes_R K(x_2) \otimes_R \cdots \otimes_R K(x_n).$$

Notation 4.5.1 If A is an R-module, we define

$$H_q(\boldsymbol{x}, A) = H_q(K(\boldsymbol{x}) \otimes_R A);$$
$$H^q(\boldsymbol{x}, A) = H^q(\mathrm{Hom}(K(\boldsymbol{x}), A)).$$

The degree p part of $K(\boldsymbol{x})$ is a free R-module generated by the symbols

$$e_{i_1} \wedge \cdots \wedge e_{i_p} = 1 \otimes \cdots \otimes 1 \otimes e_{x_{i_1}} \otimes \cdots \otimes e_{x_{i_p}} \otimes \cdots \otimes 1 \ (i_1 < \cdots < i_p).$$

In particular, $K_p(\boldsymbol{x})$ is isomorphic to the p^{th} exterior product $\Lambda^p R^n$ of R^n and has rank $\binom{n}{p}$, so $K(\boldsymbol{x})$ is often called the *exterior algebra complex.* The derivative $K_p(\boldsymbol{x}) \to K_{p-1}(\boldsymbol{x})$ sends $e_{i_1} \wedge \cdots \wedge e_{i_p}$ to $\sum (-1)^{k+1} x_{i_k} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p}$. As an example, $K(x, y)$ is the complex

$$\begin{array}{cccccccccc} & 0 & \longrightarrow & R & \xrightarrow{(x,-y)} & R^2 & \xrightarrow{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]} & R & \longrightarrow & 0. \\ \text{basis:} & & & \{e_x \wedge e_y\} & & \{e_y, e_x\} & & \{1\} & & \end{array}$$

DG-Algebras 4.5.2 A *graded R-algebra* K_* is a family $\{K_p, p \geq 0\}$ of R-modules, equipped with a bilinear product $K_p \otimes_R K_q \to K_{p+q}$ and an element $1 \in K_0$ making K_0 and $\oplus K_p$ into associative R-algebras with unit. K_* is *graded-commutative* if for every $a \in K_p$, $b \in K_q$ we have $a \cdot b = (-1)^{pq} b \cdot a$. A *differential graded algebra*, or *DG-algebra*, is a graded R-algebra K_* equipped with a map $d : K_p \to K_{p-1}$, satisfying $d^2 = 0$ and satisfying the *Leibnitz rule*:

$$d(a \cdot b) = d(a) \cdot b + (-1)^p a \cdot d(b) \text{ for } a \in K_p.$$

Exercise 4.5.1

1. Let K be a DG-algebra. Show that the homology $H_*(K) = \{H_p(K)\}$ forms a graded R-algebra, and that $H_*(K)$ is graded-commutative whenever K_* is.
2. Show that the Koszul complex $K(\boldsymbol{x}) \cong \Lambda^*(R^n)$ is a graded-commutative DG-algebra. If R is commutative, use this to obtain an external product $H_p(\boldsymbol{x}, A) \otimes_R H_q(\boldsymbol{x}, B) \to H_{p+q}(\boldsymbol{x}, A \otimes_R B)$. Conclude that if A is a commutative R-algebra then the Koszul homology $H_*(\boldsymbol{x}, A)$ is a graded-commutative R-algebra.
3. If $x_1, \cdots \in I$ and $A = {}^{R}\!/_{I}$, show that $H_*(\boldsymbol{x}, A)$ is the exterior algebra $\Lambda^*(A^n)$.

1. We must first show that $H_*(K)$ has a bilinear product $H_p(K) \otimes_R H_q(K) \to H_{p+q}(K)$ and there exists $1 \in H_0(K)$ such that $H_0(K)$ and $\bigoplus H_p(K)$ are associative unital R-algebras. Subsequently, we will show that if K_* is graded-commutative, then $H_*(K)$ is

graded commutative.

We proceed. The bilinear product $H_p(K) \otimes_R H_q(K) \to H_{p+q}(K)$ is defined by the induced map from the following bilinear construction:

$$H_p(K) \times H_q(K) \to H_{p+q}(K),$$
$$([x],[y]) \mapsto [xy].$$

We must show that this map is well-defined. If $[x] \in H_p(K) = {}^{\ker d_p}\!/_{\operatorname{im} d_{p+1}}$, then $x = x_0 + d_{p+1}(x_1)$, and similarly $y = y_0 + d_{q+1}(y_1)$. Observe that

$$\begin{aligned} xy &= \big(x_0 + d(x_1)\big)\big(y_0 + d(y_1)\big) \\ &= x_0y_0 + x_0d(y_1) + y_0d(x_1) + d(x_1)d(y_1), \text{ so} \\ xy - x_0y_0 &= x_0d(y_1) + y_0d(x_1) + d(x_1)d(y_1). \end{aligned}$$

We need to show that $x_0d(y_1) + y_0d(x_1) + d(x_1)d(y_1) \in \operatorname{im} d_{p+q+1}$, so that $[xy] \in {}^{\ker d_{p+q}}\!/_{\operatorname{im} d_{p+q+1}} = H_{p+q}(K)$. (Certainly, since K is a DG-algebra, $xy \in K_{p+q}$, and $xy \in \ker d_{p+q}$ since $d(xy) = d(x)y + (-1)^p x d(y) = 0 + 0 = 0$.) To see that $x_0d(y_1) + y_0d(x_1) + d(x_1)d(y_1) \in \operatorname{im} d_{p+q+1}$, we claim that

$$x_1y_0 + (-1)^p x_0y_1 + x_1d(y_0) \xmapsto{\;d\;} x_0d(y_1) + y_0d(x_1) + d(x_1)d(y_1),$$

so that it is in the image of d. Indeed, we may compute, using linearity and the Leibniz rule:

$$\begin{aligned} & d\big(x_1y_0 + (-1)^p x_0y_1 + x_1d(y_0)\big) \\ &= d(x_1y_0) + (-1)^p d(x_0y_1) + d\big(x_1d(y_0)\big) \\ &= d(x_1)y_0 + (-1)^{p+1}x_1d(y_0) + (-1)^p\Big[d(x_0)y_1 + (-1)^p x_0d(y_1)\Big] + d(x_1)d(y_0) + (-1)^{p+1}x_1d^2(y_0) \\ &= d(x_1)y_0 + (-1)^{p+1}x_1 \cdot 0 + (-1)^p\Big[0 \cdot y_1 + (-1)^p x_0d(y_1)\Big] + d(x_1)d(y_0) + (-1)^{p+1}x_1 \cdot 0 \\ &= d(x_1)y_0 + (-1)^p(-1)^p x_0d(y_1) + d(x_1)d(y_0) \\ &= d(x_1)y_0 + x_0d(y_1) + d(x_1)d(y_0), \end{aligned}$$

as we wished to show. Next, the unit element in $H_0(K) = {}^{\ker d_0}\!/_{\operatorname{im} d_1} = {}^{K_0}\!/_{\operatorname{im} d_1}$ is the equivalence class of $1 \in K_0$, since K is a DG-algebra. Finally, since K is an associative R-algebra, $H_0(K)$ and $\bigoplus H_p(K)$ are associative R-algebras; taking equivalence classes preserves the distributivity and associativity from K.

Finally, assume that K_* is graded-commutative, so that for all $x \in K_p$ and $y \in K_q$, $xy = (-1)^{pq} yx$. By properties of equivalence classes, for $[x] \in H_p(K)$, $[y] \in H_q(K)$,

$$[x][y] = [xy] = [(-1)^{pq} yx] = (-1)^{pq}[yx] = (-1)^{pq}[y][x],$$

so $H_*(K)$ is graded-commutative.

2. To see that $K(\boldsymbol{x}) \cong \Lambda^*(R^n)$ is a graded-commutative DG-algebra, we must show:

   (a) that $\Lambda^*(R^n)$ has a bilinear product $\Lambda^p(R^n) \otimes_R \Lambda^q(R^n) \to \Lambda^{p+q}(R^n)$ and an element $1 \in \Lambda^0(R^n)$ making $\Lambda^0(R^n)$ and $\bigoplus \Lambda^p(R^n)$ into unital associative R-algebras,

   (b) that for all $a \in \Lambda^p(R^n)$ and $b \in \Lambda^q(R^n)$, $ab = (-1)^{pq} ba$, and

   (c) that $\Lambda^*(R^n)$ has a map $d : \Lambda^p(R^n) \to \Lambda^{p-1}(R^n)$ satisfying $d^2 = 0$ and $d(ab) = d(a)b + (-1)^p a d(b)$ for all $a \in \Lambda^p(R^n)$.

   So we proceed.

   (a) The bilinear product $\Lambda^p(R^n) \otimes_R \Lambda^q(R^n) \to \Lambda^{p+q}(R^n)$ is defined via the map $\Lambda^p(R^n) \times \Lambda^q(R^n) \to \Lambda^{p+q}(R^n)$ given by wedging bases of $\Lambda^p(R^n)$ and $\Lambda^q(R^n)$:

   $$\Big(\left(e_{i_1} \wedge \cdots \wedge e_{i_p}\right), \left(e_{j_1} \wedge \cdots \wedge e_{j_q}\right) \Big) \mapsto e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_{j_1} \wedge \cdots \wedge e_{j_q}.$$

   The element $1 \in \Lambda^0(R^n) \cong R^{\binom{n}{0}} = R$ is the unit in R. Finally, $\Lambda^0(R^n) \cong R$ is an associative R-algebra trivially, and $\bigoplus \Lambda^p(R^n) \cong \bigoplus R^{\binom{n}{p}}$ is an associative R-algebra as well.

   (b) To check that $\Lambda^*(R^n)$ is graded-commutative, it suffices to check the skew-commutativity on basis elements. Let $e_{i_1} \wedge \cdots \wedge e_{i_p} \in \Lambda^p(R^n)$ and let $e_{j_1} \wedge \cdots \wedge e_{j_q} \in \Lambda^q(R^n)$. Consequently,

   $$\begin{aligned}
   \left(e_{i_1} \wedge \cdots \wedge e_{i_p}\right) \wedge \left(e_{j_1} \wedge \cdots \wedge e_{j_q}\right) &= e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge \left(e_{j_1} \wedge \cdots \wedge e_{j_q}\right) \\
   &= (-1)^q e_{i_1} \wedge \cdots \wedge \left(e_{j_1} \wedge \cdots \wedge e_{j_q}\right) \wedge e_{i_p} \\
   &= (-1)^{2q} e_{i_1} \wedge \cdots \wedge \left(e_{j_1} \wedge \cdots \wedge e_{j_q}\right) \wedge e_{i_{p-1}} \wedge e_{i_p} \\
   &\;\;\vdots \\
   &= (-1)^{(p-1)q} e_{i_1} \wedge \left(e_{j_1} \wedge \cdots \wedge e_{j_q}\right) \wedge e_{i_2} \wedge \cdots \wedge e_{i_p} \\
   &= (-1)^{pq} \left(e_{j_1} \wedge \cdots \wedge e_{j_q}\right) \wedge e_{i_1} \wedge \cdots \wedge e_{i_p},
   \end{aligned}$$

so $\Lambda^*(R^n)$ is graded-commutative, as desired.

(c) Now we must show that $d : K_p(\boldsymbol{x}) \to K_{p-1}(\boldsymbol{x})$ defined by

$$d\left(e_{i_1} \wedge \cdots \wedge e_{i_p}\right) = \sum_{k=1}^{p} (-1)^{k+1} x_{i_k} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p}$$

satisfies $d^2 = 0$ and the Leibniz rule. Note that we will not be particular about the order of the x_{i_k} elements when multiple appear; since $\boldsymbol{x} = (x_1, ..., x_n)$ is a sequence of central elements, we may commute them without worry. First, we see that $d^2 = 0$ by computing on basis elements:

$$\begin{aligned}
&d^2\left(e_{i_1} \wedge \cdots \wedge e_{i_p}\right) \\
&= d\left(\sum_{k=1}^{p} (-1)^{k+1} x_{i_k} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p}\right) \\
&= \sum_{k=1}^{p} (-1)^{k+1} x_{i_k} d\left(e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p}\right) \\
&= \sum_{k=1}^{p} (-1)^{k+1} x_{i_k} \left(\sum_{\substack{j=1 \\ j<k}}^{p} (-1)^{j+1} x_{i_j} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p}\right. \\
&\qquad\qquad \left. + \sum_{\substack{j=1 \\ j>k}}^{p} (-1)^{j} x_{i_j} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_p}\right).
\end{aligned}$$

Note that the second term has a factor of $(-1)^j$ rather than $(-1)^{j+1}$; this is because the omission of the e_{i_k} term occurs in a index lower than the omission of the e_{i_j} term, and thus throws off the parity. We continue, by distributing:

$$
\begin{aligned}
&d^2\left(e_{i_1}\wedge\cdots\wedge e_{i_n}\right)\\
&\quad=\sum_{k=1}^{p}\sum_{\substack{j=1\\ j<k}}^{p}(-1)^{k+j}x_{i_k}x_{i_j}e_{i_1}\wedge\cdots\wedge\widehat{e_{i_j}}\wedge\cdots\wedge\widehat{e_{i_k}}\wedge\cdots\wedge e_{i_p}\\
&\quad+\sum_{k=1}^{p}\sum_{\substack{j=1\\ j>k}}^{p}(-1)^{k+j+1}x_{i_k}x_{i_j}e_{i_1}\wedge\cdots\wedge\widehat{e_{i_k}}\wedge\cdots\wedge\widehat{e_{i_j}}\wedge\cdots\wedge e_{i_p}\\
&\quad=\sum_{k=1}^{p}\sum_{\substack{j=1\\ j<k}}^{p}(-1)^{k+j}x_{i_k}x_{i_j}e_{i_1}\wedge\cdots\wedge\widehat{e_{i_j}}\wedge\cdots\wedge\widehat{e_{i_k}}\wedge\cdots\wedge e_{i_p}\\
&\quad-\sum_{k=1}^{p}\sum_{\substack{j=1\\ j>k}}^{p}(-1)^{k+j}x_{i_k}x_{i_j}e_{i_1}\wedge\cdots\wedge\widehat{e_{i_k}}\wedge\cdots\wedge\widehat{e_{i_j}}\wedge\cdots\wedge e_{i_p}.
\end{aligned}
$$

Next we add up like terms with common basis elements. This results in, after reindexing,

$$
\begin{aligned}
&d^2\left(e_{i_1}\wedge\cdots\wedge e_{i_n}\right)\\
&\quad=\sum_{k=1}^{p}\sum_{\substack{j=1\\ j<k}}^{p}(-1)^{j+k}\left(x_{i_k}x_{i_j}-x_{i_k}x_{i_j}\right)e_{i_1}\wedge\cdots\wedge\widehat{e_{i_j}}\wedge\cdots\wedge\widehat{e_{i_k}}\wedge\cdots\wedge e_{i_p}\\
&\quad=0,
\end{aligned}
$$

so $d^2=0$ as desired. Next, we show that d satisfies the Leibniz rule. Again, we compute on basis elements:

$$
\begin{aligned}
d\Big(&\left(e_{i_1}\wedge\cdots\wedge e_{i_p}\right)\wedge\left(e_{j_1}\wedge\cdots\wedge e_{j_q}\right)\Big)\\
&=\sum_{k=1}^{p+q}(-1)^{k+1}x_{i_k}e_{i_1}\wedge\cdots\wedge\widehat{e_k}\wedge\cdots\wedge e_{j_q}\\
&=\left(\sum_{k=1}^{p}(-1)^{k+1}x_{i_k}e_{i_1}\wedge\cdots\wedge\widehat{e_{i_k}}\wedge\cdots\wedge e_{i_p}\right)\wedge e_{j_1}\wedge\cdots\wedge e_{j_q}\\
&+e_{i_1}\wedge\cdots\wedge e_{i_p}\wedge\left(\sum_{k=1}^{q}(-1)^{p+k+1}x_{j_k}e_{j_1}\wedge\cdots\wedge\widehat{e_{j_k}}\wedge\cdots\wedge e_{j_q}\right).
\end{aligned}
$$

The alternating sign in the second term is offset by an additional p to account for the first p terms. Subsequently,

$$\begin{aligned}
&d\Big(\left(e_{i_1} \wedge \cdots \wedge e_{i_p}\right) \wedge \left(e_{j_1} \wedge \cdots \wedge e_{j_q}\right) \Big)\\
&= \left(\sum_{k=1}^{p}(-1)^{k+1}x_{i_k}e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p}\right) \wedge e_{j_1} \wedge \cdots \wedge e_{j_q}\\
&+ e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge \left(\sum_{k=1}^{q}(-1)^{p+k+1}x_{j_k}e_{j_1} \wedge \cdots \wedge \widehat{e_{j_k}} \wedge \cdots \wedge e_{j_q}\right)\\
&= \left(\sum_{k=1}^{p}(-1)^{k+1}x_{i_k}e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p}\right) \wedge e_{j_1} \wedge \cdots \wedge e_{j_q}\\
&+ (-1)^p e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge \left(\sum_{k=1}^{q}(-1)^{k+1}x_{j_k}e_{j_1} \wedge \cdots \wedge \widehat{e_{j_k}} \wedge \cdots \wedge e_{j_q}\right)\\
&= d\left(e_{i_1} \wedge \cdots \wedge e_{i_p}\right) \wedge \left(e_{j_1} \wedge \cdots \wedge e_{j_q}\right) + (-1)^p\left(e_{i_1} \wedge \cdots \wedge e_{i_p}\right) \wedge d\left(e_{j_1} \wedge \cdots \wedge e_{j_q}\right),
\end{aligned}$$

and hence the Leibniz rule is satisfied as well.

Next, we must show that if R is commutative, then there exists an external product $H_p(\boldsymbol{x}, A) \otimes_R H_q(\boldsymbol{x}, B) \to H_{p+q}(\boldsymbol{x}, A \otimes_R B)$. Indeed, the external product is defined by the following bilinear map:

$$\Big(a\left(e_{i_1} \wedge \cdots \wedge e_{i_p}\right), b\left(e_{j_1} \wedge \cdots \wedge e_{j_q}\right)\Big) \mapsto a \otimes b\left(e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_{j_1} \wedge \cdots \wedge e_{j_q}\right).$$

Finally, we must conclude that if A is a commutative R-algebra, then $H_*(\boldsymbol{x}, A)$ is a graded-commutative R-algebra. Indeed, this follows from part 1. Any commutative R-algebra is a graded-commutative DG-algebra with trivial grading (namely, $A_0 = A$, $A_i = 0$ for $i > 0$), so by part 1., the Koszul homology is a graded-commutative R-algebra.

3. Let $x_1, \ldots \in I$ and let $A = R/I$. We must show that $H_*(\boldsymbol{x}, A)$ is the exterior algebra $\Lambda^*(A^n)$. Indeed, $H_p(\boldsymbol{x}, A)$ is defined to be $H_p(K(\boldsymbol{x}) \otimes_R A) = H_p\left(K(\boldsymbol{x}) \otimes_R R/I\right)$. Since $\boldsymbol{x} \subseteq I$,

$$\begin{aligned}
K_p(\boldsymbol{x}) &\xrightarrow{\ d\ } K_{p-1}(\boldsymbol{x})\\
e_{i_1} \wedge \cdots \wedge e_{i_p} &\longmapsto \sum_{k=1}^{p}(-1)^{k+1}x_{i_k}e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p}
\end{aligned}$$

becomes, after tensoring with $A = R/I$,

$$K_p(\boldsymbol{x}) \otimes_R {}^{R}\!/\!_{I} \xrightarrow{d \otimes \mathrm{id}_{R/I}} K_{p-1}(\boldsymbol{x}) \otimes_R {}^{R}\!/\!_{I}$$

$$\begin{aligned}
\left(e_{i_1} \wedge \cdots \wedge e_{i_p}\right) \otimes a \longmapsto & \left(\sum_{k=1}^{p} (-1)^{k+1} x_{i_k} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p}\right) \otimes a \\
&= \sum_{k=1}^{p} \left((-1)^{k+1} x_{i_k} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p} \otimes a\right) \\
&= \sum_{k=1}^{p} \left((-1)^{k+1} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p} \otimes x_{i_k} a\right) \\
&= \sum_{k=1}^{p} \left((-1)^{k+1} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p} \otimes 0\right) \\
&= 0.
\end{aligned}$$

Hence every map in $K(\boldsymbol{x}) \otimes_R A$ is the zero map, and therefore the homology is isomorphic to the Koszul complex itself, which is $\Lambda^*(A^n)$, and the result is shown.

Exercise 4.5.2 Show that $\{H_q(\boldsymbol{x}, -)\}$ is a homological δ-functor, and that $\{H^q(\boldsymbol{x}, -)\}$ is a cohomological δ-functor with

$$H_0(\boldsymbol{x}, A) = {}^{A}\!/\!_{(x_1, \cdots, x_n)A}$$
$$H^0(\boldsymbol{x}, A) = \operatorname{Hom}\left({}^{R}\!/\!_{\boldsymbol{x}R}, A\right) = \{a \in A \mid x_i a = 0 \text{ for all } i\}.$$

Then show that there are isomorphisms $H_p(\boldsymbol{x}, A) \cong H^{n-p}(\boldsymbol{x}, A)$ for all p.

We show that $\{H_q(\boldsymbol{x}, -)\} = \{H_q(K(\boldsymbol{x}) \otimes_R -)\}$ is a homological δ-functor; the proof that $\{H^q(\boldsymbol{x}, -)\}$ is a cohomological δ-functor is completely analogous. We must show that if $0 \to A \to B \to C \to 0$ is a short exact sequence of R-modules, then there exists $\delta_q : H_q(\boldsymbol{x}, C) \to H_{q-1}(\boldsymbol{x}, A)$ such that

$$\cdots \to H_{q+1}(\boldsymbol{x}, C) \xrightarrow{\delta} H_q(\boldsymbol{x}, A) \to H_q(\boldsymbol{x}, B) \to H_q(\boldsymbol{x}, C) \xrightarrow{\delta} H_{q-1}(\boldsymbol{x}, A) \to \cdots$$

is a long exact sequence, and that if

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

is a morphism of short exact sequences, then the following square commutes (giving us the commutative ladder):

$$\begin{array}{ccc} H_q(\boldsymbol{x}, C) & \xrightarrow{\delta} & H_{q-1}(\boldsymbol{x}, A) \\ \downarrow & & \downarrow \\ H_q(\boldsymbol{x}, C') & \xrightarrow{\delta} & H_{q-1}(\boldsymbol{x}, A') \end{array}$$

We show the existence of δ via the Snake Lemma. Note that for all $q \geq 0$ and for any R-module M, we have $K_q(\boldsymbol{x}) \otimes_R M \cong R^{\binom{n}{q}} \otimes_R M \cong M^{\binom{n}{q}} \cong \Lambda^q(M^n)$. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules, and consider the following diagram:

$$\begin{array}{ccccccc} & & \Lambda^q(A^n)/d\Lambda^{q+1}(A^n) & \longrightarrow & \Lambda^q(B^n)/d\Lambda^{q+1}(B^n) & \longrightarrow & \Lambda^q(C^n)/d\Lambda^{q+1}(C^n) & \longrightarrow 0 \\ & & \downarrow d & & \downarrow d & & \downarrow d & \\ 0 & \longrightarrow & Z_{q-1}\Lambda^*(A^n) & \longrightarrow & Z_{q-1}\Lambda^*(B^n) & \longrightarrow & Z_{q-1}\Lambda^*(C^n) & \end{array}$$

The squares are commutative, so we must show that the rows are exact to apply the Snake Lemma. Once this has been done, we can apply the Snake Lemma to get the following long exact sequence:

$$\begin{tikzcd} H_q(\boldsymbol{x}, A) \arrow[r] & H_q(\boldsymbol{x}, B) \arrow[r] & H_q(\boldsymbol{x}, C) \arrow[dll, "\delta"] \\ H_{q-1}(\boldsymbol{x}, A) \arrow[r] & H_{q-1}(\boldsymbol{x}, B) \arrow[r] & H_{q-1}(\boldsymbol{x}, C) \end{tikzcd}$$

The first row is exact because it is the result of tensoring over R the entries of the short exact sequence $0 \to A \to B \to C \to 0$ with $\Lambda^q(R^n)/d\Lambda^{q+1}(R^n)$. Tensoring is right exact, and hence the first row is exact.

The second row is exact by observing the following commutative diagram which includes the kernel complex into the whole complex:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_{q-1}\Lambda^*(A^n) & \longrightarrow & Z_{q-1}\Lambda^*(B^n) & \longrightarrow & Z_{q-1}\Lambda^*(C^n) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Lambda^q(A^n) & \longrightarrow & \Lambda^q(B^n) & \longrightarrow & \Lambda^q(C^n) & \longrightarrow & 0 \end{array}$$

Now, the bottom row here is exact because it is the result of tensoring over R the entries of the short exact sequence $0 \to A \to B \to C \to 0$ with $\Lambda^q(R^n) \cong R^{\binom{n}{q}}$ which is free, hence flat, and preserves left exactness as well. Now we can observe the above diagram to confirm the exactness of its top row, which will confirm our ability to use the Snake Lemma to get the connecting homomorphisms δ. Exactness at $Z_{q-1}\Lambda^*(A^n)$ follows from the fact that the map

$Z_{q-1}\Lambda^*(A^n) \to Z_{q-1}\Lambda^*(B^n)$ is a restriction of the injective map $\Lambda^q(A^n) \to \Lambda^q(B^n)$, hence injective. Exactness at $Z_{q-1}\Lambda^*(B^n)$ is done by the following diagram chase:

Let $x \in Z_{q-1}\Lambda^*(B^n)$ be in the kernel of $Z_{q-1}\Lambda^*(B^n) \to Z_{q-1}\Lambda^*(C^n)$. So by the commutativity of the right square, we have

$$\begin{tikzcd}
x \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & 0 \arrow[ddd, mapsto] \\
& Z_{q-1}\Lambda^*(B^n) \arrow[r] \arrow[d] & Z_{q-1}\Lambda^*(C^n) \arrow[d] & \\
& \Lambda^q(B^n) \arrow[r] & \Lambda^q(C^n) & \\
b \arrow[rrr, mapsto] & & & 0
\end{tikzcd}$$

By the exactness of the bottom row, since $b \in \ker\left(\Lambda^q(B^n) \to \Lambda^q(C^n)\right)$, there exists $a \in \Lambda^q(A^n)$ such that $a \mapsto b$. Now extend the first two columns into short exact sequences:

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Z_{q-1}\Lambda^*(A^n) \arrow[r] \arrow[d] & Z_{q-1}\Lambda^*(B^n) \arrow[r] \arrow[d] & Z_{q-1}\Lambda^*(C^n) \arrow[d] & \\
0 \arrow[r] & \Lambda^q(A^n) \arrow[r] \arrow[d] & \Lambda^q(B^n) \arrow[r] \arrow[d] & \Lambda^q(C^n) \arrow[r] & 0 \\
0 \arrow[r] & \Lambda^q(A^n) / Z_{q-1}\Lambda^*(A^n) \arrow[r] \arrow[d] & \Lambda^q(B^n) / Z_{q-1}\Lambda^*(B^n) \arrow[d] & & \\
& 0 & 0 & &
\end{tikzcd}$$

Turning our focus to the bottom square, we have

$$\begin{tikzcd}
a \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & b \arrow[ddd, mapsto] \\
& \Lambda^q(A^n) \arrow[r] \arrow[d] & \Lambda^q(B^n) \arrow[d] & \\
& \Lambda^q(A^n) / Z_{q-1}\Lambda^*(A^n) \arrow[r] & \Lambda^q(B^n) / Z_{q-1}\Lambda^*(B^n) & \\
{[a]} \arrow[rrr, mapsto] & & & 0
\end{tikzcd}$$

By injectivity of $\Lambda^q(A^n) / Z_{q-1}\Lambda^*(A^n) \to \Lambda^q(B^n) / Z_{q-1}\Lambda^*(B^n)$, since $[a] \mapsto 0$, $[a] = 0$, and hence a is in the kernel complex $Z_{q-1}\Lambda^*(A^n)$ and maps to x. Thus, $\ker\left(Z_{q-1}\Lambda^*(B^n) \to Z_{q-1}\Lambda^*(C^n)\right) \subseteq \operatorname{im}\left(Z_{q-1}\Lambda^*(A^n) \to Z_{q-1}\Lambda^*(B^n)\right)$.

For the other inclusion, let $y \in Z_{q-1}\Lambda^*(B^n)$ be in the image of $Z_{q-1}\Lambda^*(A^n) \to Z_{q-1}\Lambda^*(B^n)$; i.e., there exists $\alpha \in Z_{q-1}\Lambda^*(A^n)$ with $\alpha \mapsto y$. Map α to a and y to b:

$$
\begin{tikzcd}
\alpha \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & y \arrow[ddd, mapsto] \\
& Z_{q-1}\Lambda^*(A^n) \arrow[r] \arrow[d] & Z_{q-1}\Lambda^*(B^n) \arrow[d] & \\
& \Lambda^q(A^n) \arrow[r] & \Lambda^q(B^n) & \\
a \arrow[rrr, mapsto] & & & b
\end{tikzcd}
$$

By exactness of $\Lambda^q(A^n) \to \Lambda^q(B^n) \to \Lambda^q(C^n)$, $a \mapsto b \mapsto 0$, and by the commutativity of the appropriate square, we have

$$
\begin{tikzcd}
y \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & \gamma \arrow[ddd, mapsto] \\
& Z_{q-1}\Lambda^*(B^n) \arrow[r] \arrow[d] & Z_{q-1}\Lambda^*(C^n) \arrow[d] & \\
& \Lambda^q(B^n) \arrow[r] & \Lambda^q(C^n) & \\
b \arrow[rrr, mapsto] & & & 0
\end{tikzcd}
$$

Finally, since the column $0 \to Z_{q-1}\Lambda^*(C^n) \to \Lambda^q(C^n)$ is exact and $\gamma \mapsto 0$, γ must be equal to 0, and therefore $y \mapsto 0$. Hence $\operatorname{im}(Z_{q-1}\Lambda^*(A^n) \to Z_{q-1}\Lambda^*(B^n)) \subseteq \ker(Z_{q-1}\Lambda^*(B^n) \to Z_{q-1}\Lambda^*(C^n))$, proving exactness at $Z_{q-1}\Lambda^*(B^n)$.

Thus, we may conclude that the Snake Lemma hypotheses are met, and we have the connecting homomorphisms δ as desired in showing that $H_q(\boldsymbol{x}, -)$ is a homological δ-functor.

It remains to show the commutative ladder; that is, if we have a commutative diagram of R-modules

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}
$$

then the following diagram commutes.

$$
\begin{tikzcd}
H_q(\boldsymbol{x}, C) \arrow[r, "\delta"] \arrow[d] & H_{q-1}(\boldsymbol{x}, A) \arrow[d] \\
H_q(\boldsymbol{x}, C') \arrow[r, "\delta"] & H_{q-1}(\boldsymbol{x}, A')
\end{tikzcd}
$$

We work on the level of representatives of equivalence classes in the homology groups. Consider the following commutative diagram, achieved by tensoring over R the given commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}
$$

with $\Lambda^q(R^n) \cong R^{\binom{n}{q}}$, and applying the differential map to every term. On the following picture, we only draw the parts of such a diagram that will be relevant to our diagram chase that follows:

$$\begin{tikzcd}
\cdots \arrow[r] & \Lambda^q(B^n) \arrow[r] \arrow[d] \arrow[ddrrr, "d"] & \Lambda^q(C^n) \arrow[r] \arrow[d] & 0 & & \\
\cdots \arrow[r] & \Lambda^q(B'^n) \arrow[r] \arrow[ddrrr, "d"] & \Lambda^q(C'^n) \arrow[r] & 0 & & \\
& & 0 \arrow[r] & \Lambda^{q-1}(A^n) \arrow[r] \arrow[d] & \Lambda^{q-1}(B^n) \arrow[r] \arrow[d] & \cdots \\
& & 0 \arrow[r] & \Lambda^{q-1}(A'^n) \arrow[r] & \Lambda^{q-1}(B'^n) \arrow[r] & \cdots
\end{tikzcd}$$

An element $z \in H_q(\boldsymbol{x}, C)$ is represented by an element $c \in \Lambda^q(C^n)$, which maps to $c' \in \Lambda^q(C'^n)$, and c' represents the image of z, call it z', in $H_q(\boldsymbol{x}, C')$. Now since $\Lambda^q(B^n) \to \Lambda^q(C^n)$ is surjective, there exists $b \in \Lambda^q(B^n)$ that lifts c, and by the commutativity of the square

$$\begin{tikzcd}
b \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & c \arrow[ddd, mapsto] \\
& \Lambda^q(B^n) \arrow[r] \arrow[d] & \Lambda^q(C^n) \arrow[d] & \\
& \Lambda^q(B'^n) \arrow[r] & \Lambda^q(C'^n) & \\
b' \arrow[rrr, mapsto] & & & c'
\end{tikzcd}$$

the image of b, call it b', lifts c'. Apply the diagonal differential map to b'; then, db' is an element of $Z_{q-1}\Lambda^*(A'^n) \subseteq \Lambda^{q-1}(A'^n)$ and hence represents $\delta(z')$ in $H_{q-1}(\boldsymbol{x}, A')$. On the other hand, applying the differential to b results in $db \in Z_{q-1}\Lambda^*(A^n)$ representing $\delta(z) \in H_{q-1}(\boldsymbol{x}, A)$, and $\delta(z)$ must map to $\delta(z')$, since db maps to db'. Therefore, we can see that the following square commutes, as desired:

$$\begin{tikzcd}
z \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & \delta(z) \arrow[ddd, mapsto] \\
& H_q(\boldsymbol{x}, C) \arrow[r] \arrow[d] & H_{q-1}(\boldsymbol{x}, A) \arrow[d] & \\
& H_q(\boldsymbol{x}, C') \arrow[r] & H_{q-1}(\boldsymbol{x}, A') & \\
z' \arrow[rrr, mapsto] & & & \delta(z')
\end{tikzcd}$$

Hence, $\{H_q(\boldsymbol{x}, -)\}$ is a homological δ-functor, as desired. Again, the proof that $\{H^q(\boldsymbol{x}, -)\}$ is a cohomological δ-functor is similar and omitted.

•••

For the next step, we need to show that $H_0(\boldsymbol{x}, A) = {}^{A}\!\diagup_{(x_1, ..., x_n)A}$. Observe that

$$\begin{aligned} H_0(\boldsymbol{x}, A) = H_0(K(\boldsymbol{x}) \otimes_R A) &= {}^{\ker(K_0(\boldsymbol{x}) \otimes_R A \to 0)}\!\diagup_{\operatorname{im}(K_1(\boldsymbol{x}) \otimes_R A \to K_0(\boldsymbol{x}) \otimes_R A)} \\ &= {}^{K_0(\boldsymbol{x}) \otimes_R A}\!\diagup_{\operatorname{im}(K_1(\boldsymbol{x}) \otimes_R A \to K_0(\boldsymbol{x}) \otimes_R A)} \\ &= {}^{R \otimes_R A}\!\diagup_{\operatorname{im}(R^n \otimes_R A \to R \otimes_R A)} \\ &= {}^{A}\!\diagup_{\operatorname{im}(A^n \to A)}, \end{aligned}$$

so we must determine $\operatorname{im}(A^n \to A)$. Observe that for a generator $e_{i_1} \in K_1(\boldsymbol{x}) \cong \Lambda^1(R^n) \cong R^{\binom{n}{1}} = R^n$,

$$\begin{aligned} K_1(\boldsymbol{x}) \otimes_R A &\xrightarrow{d \otimes \mathrm{id}_A} K_0(\boldsymbol{x}) \otimes_R A \\ e_{i_1} \otimes a &\longmapsto \sum_{k=1}^{1} (-1)^{k+1} x_{i_k} \widehat{e_{i_k}} \otimes a \\ &\qquad = x_{i_1} \otimes a \end{aligned}$$

The isomorphism $K_0(\boldsymbol{x}) \otimes_R A \cong A$ takes $x_{i_1} \otimes a$ to $x_{i_1} a$, so $\operatorname{im}(d \otimes \mathrm{id}_A) = \boldsymbol{x}A$, so that $H_0(\boldsymbol{x}, A) \cong {}^{A}\!\diagup_{(x_1, ..., x_n)A}$, as desired.

Next, we want to show that $H^0(\boldsymbol{x}, A) = \operatorname{Hom}\left({}^{R}\!\diagup_{\boldsymbol{x}R}, A\right) = \{a \in A \mid x_i a = 0 \text{ for all } i\}$. The isomorphism $\operatorname{Hom}\left({}^{R}\!\diagup_{\boldsymbol{x}R}, A\right) \cong \{a \in A \mid x_i a = 0\}$ is clear via the map

$$\begin{aligned} \operatorname{Hom}\left({}^{R}\!\diagup_{\boldsymbol{x}R}, A\right) &\to \{a \in A \mid x_i a = 0\} \\ f &\mapsto f([1]) \end{aligned}$$

because $x_i f([1]) = f(x_i[1]) = f(0) = 0$, and it has inverse

$$\begin{aligned} \{a \in A \mid x_i a = 0\} &\to \operatorname{Hom}\left({}^{R}\!\diagup_{\boldsymbol{x}R}, A\right) \\ a &\mapsto g \text{ such that } g([1]) = a. \end{aligned}$$

Indeed,

$$\begin{aligned}\operatorname{Hom}\left({}^{R}\!/\!_{\boldsymbol{x}R}, A\right) &\longrightarrow \{a \in A \mid x_i a = 0\} \longrightarrow \operatorname{Hom}\left({}^{R}\!/\!_{\boldsymbol{x}R}, A\right)\\ f &\longmapsto f([1]) \longmapsto f\end{aligned}$$

and

$$\begin{aligned}\{a \in A \mid x_i a = 0\} &\longrightarrow \operatorname{Hom}\left({}^{R}\!/\!_{\boldsymbol{x}R}, A\right) \longrightarrow \{a \in A \mid x_i a = 0\}\\ a &\longmapsto g \longmapsto a.\end{aligned}$$

Now observe that

$$\begin{aligned}H^0(\boldsymbol{x}, A) &= H^0(\operatorname{Hom}(K(\boldsymbol{x}), A))\\ &= {}^{\ker(\operatorname{Hom}(K_0(\boldsymbol{x}), A) \to \operatorname{Hom}(K_1(\boldsymbol{x}), A))}\!\Big/\!_{\operatorname{im}(\operatorname{Hom}(0, A) \to \operatorname{Hom}(K_0(\boldsymbol{x}), A))}\\ &= \ker\big(\operatorname{Hom}(K_0(\boldsymbol{x}), A) \to \operatorname{Hom}(K_1(\boldsymbol{x}), A)\big).\end{aligned}$$

See that $\operatorname{Hom}(K_0(\boldsymbol{x}), A) \to \operatorname{Hom}(K_1(\boldsymbol{x}), A)$ is defined via

$$\begin{tikzcd}
e_{i_1} \arrow[rr, mapsto] & & x_{i_1} \arrow[dd, mapsto] \\
K_1(\boldsymbol{x}) \arrow[r, "d"] & K_0(\boldsymbol{x}) \arrow[d, "f"] & \\
& A & f(x_{i_1})
\end{tikzcd}$$

Note that since $K_0(\boldsymbol{x}) \cong R$, f is determined by the image of 1 in A. If $f(1) = a$, then $\ker\big(\operatorname{Hom}(K_0(\boldsymbol{x}), A) \to \operatorname{Hom}(K_1(\boldsymbol{x}), A)\big)$ is $\{a = f(1) \in A \mid 0 = f(x_{i_1}) = x_{i_1} f(1) = x_{i_1} a\}$, as desired.

• • •

Finally, we must show that $H_p(\boldsymbol{x}, A) \cong H^{n-p}(\boldsymbol{x}, A)$ for all $p \in \{0, ..., n\}$. Indeed, first notice that $K_n(\boldsymbol{x}) \cong \Lambda^n(R^n) \cong R^{\binom{n}{n}} = R$ under the explicit isomorphism

$$\begin{aligned}\omega_n : K_n(\boldsymbol{x}) &\longrightarrow R\\ e_{i_1} \wedge \cdots \wedge e_{i_n} &\longmapsto 1.\end{aligned}$$

Write $(K_\ell(\boldsymbol{x}))^*$ for the dual of $K_\ell(\boldsymbol{x})$; i.e., $(K_\ell(\boldsymbol{x}))^* = \operatorname{Hom}_R(K_\ell(\boldsymbol{x}), R)$. Notice that $R \cong \operatorname{Hom}(R, R) = \operatorname{Hom}\left(R^{\binom{n}{0}}, R\right) \cong \operatorname{Hom}(K_0(\boldsymbol{x}), R) = (K_0(\boldsymbol{x}))^*$. Thus the above map is $\omega_n : K_n(\boldsymbol{x}) \to (K_0(\boldsymbol{x}))^*$, and this generalizes; we can then define maps

$$\begin{aligned}\omega_i : K_p(\boldsymbol{x}) &\longrightarrow (K_{n-p}(\boldsymbol{x}))^*\\ x &\longmapsto (\omega_p(x))(y) = \omega_n(x \wedge y)\end{aligned}$$

for $x \in K_p(\boldsymbol{x})$ and $y \in K_{n-p}(\boldsymbol{x})$. Now see that for generators $e_{j_1} \wedge \cdots \wedge e_{j_p} \in K_p(\boldsymbol{x})$ and $e_{k_1} \wedge \cdots \wedge e_{k_{n-p}} \in K_{n-p}(\boldsymbol{x})$, we have

$$\begin{aligned}
&(\omega_p(e_{j_1} \wedge \cdots \wedge e_{j_p}))(e_{k_1} \wedge \cdots \wedge e_{k_{n-p}}) \\
&\qquad = \omega_n(e_{j_1} \wedge \cdots \wedge e_{j_p} \wedge e_{k_1} \wedge \cdots \wedge e_{k_{n-p}}) \\
&\qquad = \begin{cases} 0 & \text{if } e_{j_\ell} = e_{k_m} \text{ for some } j_\ell \neq k_m \\ \omega_n((-1)^\kappa e_{i_1} \wedge \cdots \wedge e_{i_n}) = (-1)^\kappa & \text{else,} \end{cases}
\end{aligned}$$

where κ is the number of times elements e_{j_ℓ} and e_{k_m} needed to commute to put

$$e_{j_1} \wedge \cdots \wedge e_{j_p} \wedge e_{k_1} \wedge \cdots \wedge e_{k_{n-p}}$$

in the order $e_{i_1} \wedge \cdots \wedge e_{i_n}$, the basis element of $K_n(\boldsymbol{x})$. Hence ω_i takes generators $e_{j_1} \wedge \cdots \wedge e_{j_p}$ of $K_p(\boldsymbol{x})$ to generators $(-1)^\kappa (e_{k_1} \wedge \cdots \wedge e_{k_{n-p}})^*$ on $(K_{n-p}(\boldsymbol{x}))^*$, and thus $\omega_i : K_p(\boldsymbol{x}) \to (K_{n-p}(\boldsymbol{x}))^*$ is an isomorphism.

For our next step, consider the diagram

$$\begin{tikzcd}
K(\boldsymbol{x}): & 0 \arrow[r] & K_n(\boldsymbol{x}) \arrow[r, "d_n"] \arrow[d, "\omega_n", "\wr"'] & K_{n-1}(\boldsymbol{x}) \arrow[r, "d_{n-1}"] \arrow[d, "\omega_{n-1}", "\wr"'] & \cdots \arrow[r, "d_2"] & K_1(\boldsymbol{x}) \arrow[r, "d_1"] \arrow[d, "\omega_1", "\wr"'] & K_0(\boldsymbol{x}) \arrow[r] \arrow[d, "\omega_0", "\wr"'] & 0 \\
(K(\boldsymbol{x}))^*: & 0 \arrow[r] & (K_0(\boldsymbol{x}))^* \arrow[r, "(d_1)^*"] & (K_1(\boldsymbol{x}))^* \arrow[r, "(d_2)^*"] & \cdots \arrow[r, "(d_{n-1})^*"] & (K_{n-1}(\boldsymbol{x}))^* \arrow[r, "(d_n)^*"] & (K_n(\boldsymbol{x}))^* \arrow[r] & 0
\end{tikzcd}$$

where $(d_\bullet)^* = \operatorname{Hom}(d_\bullet, R)$. We claim the following:

1. The squares above commute up to sign; i.e., $\omega_{p-1} d_p = (-1)^{p-1}(d_{n-p+1})^* \omega_p$ for every $p \in \{0, ..., n\}$.
2. $K(\boldsymbol{x}) \cong (K(\boldsymbol{x}))^*$ as complexes.
3. If A is an R-module, then $K(\boldsymbol{x}) \otimes_R A \cong \operatorname{Hom}(K(\boldsymbol{x}), A)$.
4. Hence, $H_p(x, A) = H_p(K(\boldsymbol{x}) \otimes_R A) \cong H^{n-p}(\operatorname{Hom}(K(\boldsymbol{x}), A)) = H^{n-p}(x, A)$, as we needed to show.

These four steps will give us the desired result. We proceed:

1. Fix p. We must show $\omega_{p-1} d_p = (-1)^{p-1}(d_{n-p+1})^* \omega_p$. We will do so on generators. Let

$e_{j_1} \wedge \cdots \wedge e_{j_p} \in K_p(\boldsymbol{x})$. Let $e_{k_1} \wedge \cdots \wedge e_{k_{n-p}} \in K_{n-p}(\boldsymbol{x})$. Observe that

$$\begin{aligned}
\omega_{p-1} d_p(e_{j_1} \wedge \cdots \wedge e_{j_p}) &= \omega_{p-1} \left(\sum_{\ell=1}^{p} x_{j_\ell} e_{j_1} \wedge \cdots \wedge \widehat{e_{j_\ell}} \wedge \cdots \wedge e_{j_p} \right) \\
&= \sum_{\ell=1}^{p} x_{j_\ell} \omega_{p-1} \left(e_{j_1} \wedge \cdots \wedge \widehat{e_{j_\ell}} \wedge \cdots \wedge e_{j_p} \right) \\
&= \sum_{\ell=1}^{p} x_{j_\ell} (-1)^{\kappa+p-1} (e_{k_1} \wedge \cdots \wedge e_{k_{n-p}} \wedge e_{j_\ell})^*,
\end{aligned}$$

since the missing e_{j_ℓ} term means there are an additional $p-1$ terms to commute past, while

$$\begin{aligned}
(-1)^{p-1} (d_{n-p+1})^* \omega_p (e_{j_1} \wedge \cdots \wedge e_{j_p}) &= (-1)^{p-1} (d_{n-p+1})^* \left((-1)^{\kappa} (e_{k_1} \wedge \cdots \wedge e_{k_{n-p}})^* \right) \\
&= (-1)^{p-1+\kappa} (d_{n-p+1})^* \left((e_{k_1} \wedge \cdots \wedge e_{k_{n-p}})^* \right) \\
&= (-1)^{p-1+\kappa} \sum_{\ell=1}^{p} x_{j_\ell} (e_{k_1} \wedge \cdots \wedge e_{k_{n-p}} \wedge e_{j_\ell})^*.
\end{aligned}$$

Thus the squares commute up to sign, as desired.

2. Since $\omega_p : K_p(\boldsymbol{x}) \to (K_{n-p}(\boldsymbol{x}))^*$ is an isomorphism, letting $\widetilde{\omega_p} = (-1)^{\frac{p(p-1)}{2}} \omega_p$ fixes the sign issue and thus gives the desired isomorphism of complexes.

3. Let M and N be R-modules, $N \cong R^\alpha$ for some $\alpha \in \mathbf{N}$. There is an isomorphism $N^* \otimes M \cong \operatorname{Hom}(N, M)$. It is defined as follows:

   Since $N \cong R^\alpha$, N has basis $\{b_1, ..., b_\alpha\}$, and N^* has basis $\{{b_1}^*, ..., {b_\alpha}^*\}$, where ${b_i}^*$ satisfies

$$ {b_i}^*(n) = {b_i}^*(c_1 b_1 + \cdots + c_\alpha b_\alpha) = c_i $$

   for $i \in \{1, ..., \alpha\}$. Let $\sigma : N^* \otimes M \to \operatorname{Hom}(N, M)$ be defined by $\sigma(n^* \otimes m)(n) = n^*(n) \cdot m$, and let $\tau : \operatorname{Hom}(N, M) \to N^* \otimes M$ be defined by $\tau(f) = \displaystyle\sum_{i=1}^{\alpha} {b_i}^* \otimes f(b_i)$. To see that σ and τ are inverses, observe that

$$ \sigma\tau(f)(n) = \sigma\left(\sum_{i=1}^{\alpha} {b_i}^* \otimes f(b_i) \right)(n) = \sum_{i=1}^{\alpha} \sigma({b_i}^* \otimes f(b_i))(n) = \sum_{i=1}^{\alpha} {b_i}^*(n) \cdot f(b_i) = \sum_{i=1}^{\alpha} c_i \cdot f(b_i) = f\left(\sum_{i=1}^{\alpha} c_i b_i \right) = f(n), $$

   and

$$ \tau\sigma(n^* \otimes m) = \tau(n^* \cdot m) = \sum_{i=1}^{\alpha} {b_i}^* \otimes n^*(b_i) \cdot m = \sum_{i=1}^{\alpha} {b_i}^* \otimes {c_i}^* \cdot m = \sum_{i=1}^{\alpha} {b_i}^* {c_i}^* \otimes m = \left(\sum_{i=1}^{\alpha} {b_i}^* {c_i}^* \right) \otimes m = n^* \otimes m, $$

   hence $N^* \otimes M \cong \operatorname{Hom}(N, M)$, as desired. Now, since $K_\bullet(\boldsymbol{x}) \cong R^\alpha$ for $\alpha \in \mathbf{N}$, we have $(K(\boldsymbol{x}))^* \otimes A \cong \operatorname{Hom}(K(\boldsymbol{x}), A)$, and by part 2, $(K(\boldsymbol{x}))^* \cong K(\boldsymbol{x})$. Therefore, for every

R-module A,

$$K(\boldsymbol{x}) \otimes_R A \cong (K(\boldsymbol{x}))^* \otimes_R A \cong \operatorname{Hom}(K(\boldsymbol{x}), A),$$

as desired.

4. Since the complexes are isomorphic by part 3, and the differentials commute with the isomorphism by parts 1. and 2., $H_p(\boldsymbol{x}, A) \cong H^{n-p}(\boldsymbol{x}, A)$, as we yearned to demonstrate.

Lemma 4.5.3 (Künneth formula for Koszul complexes) *If $C = C_*$ is a chain complex of R-modules and $x \in R$, there are exact sequences*

$$0 \to H_0(x, H_q(C)) \to H_q(K(x) \otimes_R C) \to H_1(x, H_{q-1}(C)) \to 0.$$

Proof. Considering R as a complex concentrated in degree zero, there is a short exact sequence of complexes $0 \to R \to K(x) \to R[-1] \to 0$. Tensoring with C yields a short exact sequence of complexes whose homology long exact sequence is

$$H_{q+1}(C[-1]) \xrightarrow{\partial} H_q(C) \to H_q(K(x) \otimes C) \to H_q(C[-1]) \xrightarrow{\partial} H_{q-1}(C).$$

Identifying $H_{q+1}(C[-1])$ with $H_q(C)$, the map ∂ is multiplication by x (check this!), whence the result. □

Exercise 4.5.3 If x is a nonzerodivisor on R, that is, $H_1(K(x)) = 0$, use the Künneth formula for complexes 3.6.3 to give another proof of this result.

We must show:

Let $C = C_\bullet$ be an arbitrary chain complex of R-modules. Let $x \in R$ be not a zero divisor. Show that

$$0 \to H_0(x, H_q(C)) \to H_q(K(x) \otimes_R C) \to H_1(x, H_{q-1}(C)) \to 0$$

is a short exact sequence, using Theorem 3.6.3.

By Theorem 3.6.3, the Künneth formula for complexes, we have the exact sequence

$$0 \to \bigoplus_{r+s=q} H_r(K(x)) \otimes H_s(C) \to H_q(K(x) \otimes C) \to \bigoplus_{r+s=q-1} \operatorname{Tor}_1^R(H_r(K(x)), H_s(C)) \to 0.$$

Since $K(x)$ is the complex $0 \to R \xrightarrow{x} R \to 0$, we have

$$H_0(K(x)) = {}^{\ker(R \to 0)}\!\big/_{\operatorname{im}(R \xrightarrow{x} R)} \cong {}^{R}\!\big/_{xR},$$

$$H_1(K(x)) = {}^{\ker(R \xrightarrow{x} R)}\!\big/_{\operatorname{im}(0 \to R)} \cong \ker(R \xrightarrow{x} R) = \{r \in R \mid xr = 0\} = 0,$$

since x is not a zero divisor, and $H_i(K(x)) = 0$ for all $i > 1$. Thus the above short exact sequence simplifies to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_0(K(x)) \otimes H_q(C) & \longrightarrow & H_q(K(x) \otimes C) & \longrightarrow & \operatorname{Tor}_1^R(H_0(K(x)), H_{q-1}(C)) & \longrightarrow & 0 \\ & & \parallel & & \parallel & & \parallel & & \\ 0 & \longrightarrow & R/_{xR} \otimes H_q(C) & \longrightarrow & H_q(K(x) \otimes C) & \longrightarrow & \operatorname{Tor}_1^R\left(R/_{xR}, H_{q-1}(C)\right) & \longrightarrow & 0 \end{array}$$

as all other terms are 0. Next, $R/_{xR}$ has the projective (indeed, free) resolution $P_\bullet \to R/_{xR} \to 0$ given by $0 \to R \xrightarrow{x} R \to R/_x \to 0$, so

$$\operatorname{Tor}_1^R\left(R/_{xR}, H_{q-1}(C)\right) = H_1(P_\bullet \otimes H_{q-1}(C)) \cong H_1(K(x) \otimes H_{q-1}(C)) = H_1(x, H_{q-1}(C)),$$

since $P_\bullet \otimes H_{q-1}(C)$ is

$$0 \to R \otimes H_{q-1}(C) \xrightarrow{\cdot x \otimes \mathrm{id}_{H_{q-1}(C)}} R \otimes H_{q-1}(C) \to 0$$

and $K(x) \otimes H_{q-1}(C)$ is also

$$0 \to R \otimes H_{q-1}(C) \xrightarrow{\cdot x \otimes \mathrm{id}_{H_{q-1}(C)}} R \otimes H_{q-1}(C) \to 0,$$

and the degrees coincide, so the homologies agree. Finally,

$R/_{xR} \otimes H_q(C) \cong \operatorname{Tor}_0^R\left(R/_{xR}, H_q(C)\right) = H_0(P_\bullet \otimes H_q(C)) \cong H_0(K(x) \otimes H_q(C)) = H_0(x, H_q(C)),$

as again, the complexes $P_\bullet \otimes H_q(C)$ and $K(x) \otimes H_q(C)$ are the same. Therefore,

$$0 \to H_0(x, H_q(C)) \to H_q(K(x) \otimes_R C) \to H_1(x, H_{q-1}(C)) \to 0$$

is exact, as desired.

Exercise 4.5.4 Show that if one of the x_i is a unit of R, then the complex $K(\boldsymbol{x})$ is split exact. Deduce that in this case $H_*(\boldsymbol{x}, A) = H^*(\boldsymbol{x}, A) = 0$ for all modules A.

Let $x_{i_j} \in R$ be a unit. To show that $K(\boldsymbol{x})$ is split exact, it is equivalent to show that $\mathrm{id}_{K(\boldsymbol{x})}$ is nulhomotopic; i.e., $\mathrm{id} = ds + sd$ for some chain contraction $\{s_p : K_p(\boldsymbol{x}) \to K_{p+1}(\boldsymbol{x})\}$. Indeed, such a chain contraction is defined on basis elements by

$$s_p(e_{i_1} \wedge \cdots \wedge e_{i_p}) = {x_{i_j}}^{-1} e_{i_1} \wedge \cdots \wedge e_{i_j} \wedge \cdots \wedge e_{i_p},$$

for then

$$\begin{aligned}
&(d_{p+1}s_p + s_{p-1}d_p)(e_{i_1} \wedge \cdots \wedge e_{i_p}) \\
&= ds(e_{i_1} \wedge \cdots \wedge e_{i_p}) + sd(e_{i_1} \wedge \cdots \wedge e_{i_p}) \\
&= d\left({x_{i_j}}^{-1} e_{i_1} \wedge \cdots \wedge e_{i_j} \wedge \cdots \wedge e_{i_p}\right) + s\left(\sum_{k=1}^{p}(-1)^{k+1} x_{i_k} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p}\right) \\
&= {x_{i_j}}^{-1} d\left(e_{i_1} \wedge \cdots \wedge e_{i_j} \wedge \cdots \wedge e_{i_p}\right) + \sum_{k=1}^{p}(-1)^{k+1} x_{i_k} s\left(e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p}\right) \\
&= {x_{i_j}}^{-1}\left(x_{i_j} e_{i_1} \wedge \cdots \wedge e_{i_p} + \sum_{k=1}^{p}(-1)^{k+1} x_{i_k} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_j} \wedge \cdots \wedge e_{i_p}\right) \\
&\qquad\qquad + \sum_{k=1}^{p}(-1)^{k} x_{i_k} {x_{i_j}}^{-1} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_j} \wedge \cdots \wedge e_{i_p} \\
&= {x_{i_j}}^{-1} x_{i_j} e_{i_1} \wedge \cdots \wedge e_{i_p} + \sum_{k=1}^{p}(-1)^{k+1} {x_{i_j}}^{-1} x_{i_k} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_j} \wedge \cdots \wedge e_{i_p} \\
&\qquad\qquad - \sum_{k=1}^{p}(-1)^{k+1} {x_{i_j}}^{-1} x_{i_k} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_j} \wedge \cdots \wedge e_{i_p} \\
&= e_{i_1} \wedge \cdots \wedge e_{i_p} \\
&= \mathrm{id}_p(e_{i_1} \wedge \cdots \wedge e_{i_p}),
\end{aligned}$$

as we wished to show. We can deduce that $H_*(\boldsymbol{x}, A) = H^*(\boldsymbol{x}, A) = 0$ for all A because additive functors preserve split exact sequences, and $H_*(\boldsymbol{x}, A) = H_*(K(\boldsymbol{x}) \otimes A)$, $H^*(\boldsymbol{x}, A) = H^*(\mathrm{Hom}(K(\boldsymbol{x}), A))$ are the homologies of exact complexes under the additive functors $- \otimes A$ and $\mathrm{Hom}(-, A)$, hence acyclic themselves so have vanishing homology.

Corollary 4.5.4 (Acyclicity) *If $\boldsymbol{x}$ is a regular sequence on an R-module A, then $H_q(\boldsymbol{x}, A) = 0$ for $q \neq 0$ and $H_0(\boldsymbol{x}, A) = A/\boldsymbol{x}A$, where $\boldsymbol{x}A = (x_1, \cdots, x_n)A$.*

Proof. Since x is a nonzerodivisor on A, the result is true for $n = 1$. Inductively, letting $x = x_n$, $\boldsymbol{y} = (x_1, \cdots, x_{n-1})$, and $C = K(\boldsymbol{y}) \otimes A$, $H_q(C) = 0$ for $q \neq 0$ and $K(x) \otimes H_0(C)$ is the complex

$$0 \to A/\boldsymbol{y}A \xrightarrow{x} A/\boldsymbol{y}A \to 0.$$

The result follows from 4.5.3, since x is a nonzerodivisor on $A/\boldsymbol{y}A$. □

Corollary 4.5.5 (Koszul resolution) *If $\boldsymbol{x}$ is a regular sequence in R, then $K(\boldsymbol{x})$ is a free resolution of ${}^{R}\!\diagup_{I}$, $I = (x_1, \cdots, x_n)R$. That is, the following sequence is exact:*

$$0 \to \Lambda^n(R^n) \to \cdots \to \Lambda^2(R^n) \to R^n \xrightarrow{\boldsymbol{x}} R \to {}^{R}\!\diagup_{I} \to 0.$$

In this case we have

$$\operatorname{Tor}_p^R\left({}^{R}\!\diagup_{I}, A\right) = H_p(\boldsymbol{x}, A);$$
$$\operatorname{Ext}_R^p\left({}^{R}\!\diagup_{I}, A\right) = H^p(\boldsymbol{x}, A).$$

Exercise 4.5.5 If $\boldsymbol{x}$ is a regular sequence in R, show that the external and internal products for Tor (2.7.8 and exercise 2.7.5(4)) agree with the external and internal products for $H_*(\boldsymbol{x}, A)$ constructed in this section.

Recall that the external product for Tor is

$$\begin{aligned}
\operatorname{Tor}_p(A, B) \otimes \operatorname{Tor}_q(A', B') &= H_p(P_\bullet \otimes B) \otimes H_q(P'_\bullet \otimes B') \\
&\to H_{p+q}(\operatorname{Tot}(P_\bullet \otimes B \otimes P'_\bullet \otimes B')) \\
&\cong H_{p+q}(\operatorname{Tot}(P_\bullet \otimes P'_\bullet \otimes B \otimes B')) \\
&\cong H_{p+q}(\operatorname{Tot}(P_\bullet \otimes P'_\bullet) \otimes B \otimes B') \\
&\to H_{p+q}(P''_\bullet \otimes B \otimes B') \\
&= \operatorname{Tor}_{p+q}(A \otimes A', B \otimes B'),
\end{aligned}$$

if $P_\bullet \to A$, $P'_\bullet \to A'$, and $P''_\bullet \to A \otimes A'$ are projective resolutions. The external product for $H_*(\boldsymbol{x}, A)$ is, from Exercise 4.5.1 part 2,

$$\begin{aligned}
H_p(\boldsymbol{x}, A) \otimes H_q(\boldsymbol{x}, A) &= H_p(K(\boldsymbol{x}) \otimes A) \otimes H_q(K(\boldsymbol{x}) \otimes A) \\
&\to H_{p+q}(K(\boldsymbol{x}) \otimes A) \\
&= H_{p+q}(\boldsymbol{x}, A),
\end{aligned}$$

given by $a(e_{i_1} \wedge \cdots \wedge e_{i_p}) \cdot b(e_{j_1} \wedge \cdots \wedge e_{j_q}) = a \otimes b(e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_{j_1} \wedge \cdots \wedge e_{j_q})$. Notice that

$$\begin{aligned} H_p(K(\boldsymbol{x}) \otimes A) \otimes H_q(K(\boldsymbol{x}) \otimes A) &\to H_{p+q}(\mathrm{Tot}(K(\boldsymbol{x}) \otimes A \otimes K(\boldsymbol{x}) \otimes A)) \\ &\cong H_{p+q}(\mathrm{Tot}(K(\boldsymbol{x}) \otimes K(\boldsymbol{x}) \otimes A \otimes A)) \\ &\cong H_{p+q}(\mathrm{Tot}(K(\boldsymbol{x}) \otimes K(\boldsymbol{x})) \otimes A \otimes A) \\ &\cong H_{p+q}(\mathrm{Tot}(K(\boldsymbol{x})) \otimes A) \\ &\cong H_{p+q}(K(\boldsymbol{x}) \otimes A) \\ &= H_{p+1}(\boldsymbol{x}, A) \end{aligned}$$

gives the same map, since $M \otimes M \cong M$ for all M

Exercise 4.5.6 Let R be a regular local ring with residue field k. Show that

$$\mathrm{Tor}_p^R(k,k) \cong \mathrm{Ext}_R^p(k,k) \cong \Lambda^p k^n \cong k^{\binom{n}{p}}, \text{ where } n = \dim(R).$$

Conclude that $id_R(k) = \dim(R)$ and that as rings $\mathrm{Tor}_*^R(k,k) \cong \Lambda^*(k^n)$.

Let $(R, \mathfrak{m}, k)$ be a regular local ring. We already know that $\Lambda^p k^n \cong k^{\binom{n}{p}}$. For the other isomorphisms, since a basis of $\mathfrak{m}$ is a regular sequence by Exercise 4.4.2, we can apply Corollary 4.5.5 and Exercise 4.5.2 to get that

$$\begin{aligned} \mathrm{Tor}_p^R(k,k) &= \mathrm{Tor}_p^R\left(R/\mathfrak{m}, k\right) && \text{by definition,} \\ &= H_p(\boldsymbol{x}, k) && \text{by Corollary 4.5.5,} \\ &\cong H^{n-p}(\boldsymbol{x}, k) && \text{by Exercise 4.5.2,} \\ &= \mathrm{Ext}_R^{n-p}\left(R/\mathfrak{m}, k\right) && \text{by Corollary 4.5.5,} \\ &= \mathrm{Ext}_R^{n-p}(k,k) && \text{by definition.} \end{aligned}$$

Now all that is needed to show is $\mathrm{Tor}_p^R(k,k) \cong k^{\binom{n}{p}}$ because then $\mathrm{Ext}_R^{n-p}(k,k) \cong \mathrm{Tor}_p^R(k,k) \cong k^{\binom{n}{p}} = k^{\binom{n}{n-p}}$ for all p.

So to see that $\mathrm{Tor}_p^R(k,k) \cong k^{\binom{n}{p}}$, observe that using the free resolution $\Lambda^* R^n \to k \to 0$, we have

$$\begin{aligned}\mathrm{Tor}_p(k,k) = H_p(\Lambda^* R^n \otimes k) \cong H_p(\Lambda^* k^n) &= \ker(\Lambda^p k^n \to \Lambda^{p-1}k^n) \big/ \mathrm{im}(\Lambda^{p+1}k^n \to \Lambda^p k^n) \\ &\cong \ker\left(k^{\binom{n}{p}} \to k^{\binom{n}{p-1}}\right) \Big/ \mathrm{im}\left(k^{\binom{n}{p+1}} \to k^{\binom{n}{p}}\right) \\ &\cong k^{\binom{n}{p}} \big/ 0 \\ &= k^{\binom{n}{p}},\end{aligned}$$

since after tensoring by k, all maps become the zero map. This immediately implies the equivalence of rings $\mathrm{Tor}_*^R(k,k) \cong \Lambda^* k^n$.

We can conclude that $id_R(k) = \dim(R) = n$ because by Theorem 4.4.9, $\mathrm{Ext}_R^q(k,k) = k^{\binom{n}{q}} = 0$ for all $q > n$ implies $id_R(k) \leq n$, and on the other hand, $\mathrm{Ext}_R^n(k,k) = k^{\binom{n}{n}} = k$ implies $id(k) = n$.

Application 4.5.6 (Scheja-Storch) Here is a computation proof of Hilbert's Syzygy Theorem 4.3.8. Let F be a field, and set $R = F[x_1, \cdots, x_n]$, $S = R[y_1, \cdots, y_n]$. Let $\boldsymbol{t}$ be the sequence $(t_1, \cdots, t_n)$ of elements $t_i = y_i - x_i$ of S. Since $S = R[t_1, \cdots, t_n]$, $\boldsymbol{t}$ is a regular sequence, and $H_0(\boldsymbol{t}, S) \cong R$, so the augmented Koszul complex of $K(\boldsymbol{t})$ is exact:

$$0 \to \Lambda^n S^n \to \Lambda^{n-1} S^n \to \cdots \to \Lambda^2 S^n \to S^n \xrightarrow{\boldsymbol{t}} S \to R \to 0.$$

Since each $\Lambda^p S^n$ is a free R-module, this is in fact a split exact sequence of R-modules. Hence applying $\otimes_R A$ yields an exact sequence of every R-module A. That is, each $K(\boldsymbol{t}) \otimes_R A$ is an S-module resolution of A. Set $R' = F[y_1, \cdots, y_n]$, a subring of S. Since $t_i = 0$ on A, we may identify the R-module structure on A with the R'-module structure on A. But $S \otimes_R A \cong R' \otimes_F A$ is a free R'-module because F is a field. Therefore each $\Lambda^p S^n \otimes_R A$ is a free R'-module, and $K(\boldsymbol{t}) \otimes_R A$ is a canonical, natural resolution of A by free R'-modules. Since $K(\boldsymbol{t}) \otimes_R A$ has length n, this proves that

$$pd_R(A) = pd_{R'}(A) \leq n$$

for every R-module A. On the other hand, since $\mathrm{Tor}_n^R(F,F) \cong F$, we see that $pd_R(F) = n$. Hence the ring $R = F[x_1, \cdots, x_n]$ has global dimension n.

4.6 Local Cohomology

Definition 4.6.1 If I is a finitely generated ideal in a commutative ring R and A is an R-module, we define

$$H_I^0(A) = \{a \in A \mid (\exists i) I^i a = 0\} = \varinjlim \mathrm{Hom}\left(R/I^i, A\right).$$

Since each $\mathrm{Hom}\left(R/I^i, -\right)$ is left exact and $\varinjlim$ is exact, we see that H_I^0 is an additive left exact functor from R-**mod** to itself. We set

$$H_I^q(A) = (R^q H_I^0)(A).$$

Since the direct limit is exact, we also have

$$H_I^q(A) = \varinjlim \mathrm{Ext}_R^q\left(R/I^i, A\right).$$

Exercise 4.6.1 Show that if $J \subseteq I$ are finitely generated ideals such that $I^i \subseteq J$ for some i, then $H^q_J(A) \cong H^q_I(A)$ for all R-modules A and all q.

We follow Lance's suggestion and use the fact that

$$H^q_I(A) = R^q H^0_I(A) = R^q\{a \in A \mid I^k a = 0 \text{ for some } k\}.$$

We show that $H^0_I = H^0_J$; hence, the derived functors coincide for all q as well. Note first that if $\mathfrak{i} \subseteq \mathfrak{j}$, then $\mathfrak{i}^n \subseteq \mathfrak{j}^n$ for all n, since an element in $\mathfrak{i}^n$ is a finite sum $\sum i_1 \cdots i_n$ for $i_1, ..., i_n \in \mathfrak{i}$, and such an element is an element of $\mathfrak{j}^n$ since $i_1, ..., i_n \in \mathfrak{i} \subseteq \mathfrak{j}$. Note second that if $\mathfrak{i}a = 0$, then for every element $\iota \in \mathfrak{i}$, $\iota \cdot a = 0$, so if $\mathfrak{h} \subseteq \mathfrak{i}$, then $\mathfrak{h}a = 0$ as well.

Let A be any R-module. For arbitrary $a \in H^0_I(A)$, $I^k a = 0$ for some k. Since $J \subseteq I$ by hypothesis, $J^k \subseteq I^k$ by above note 1, so $J^k a = 0$ by above note 2, and thus $a \in H^0_J(A)$.

On the other hand, let $a \in H^0_J(A)$, so that $J^k a = 0$ for some k. Since $I^i \subseteq J$ by hypothesis, $(I^i)^k \subseteq J^k$ by above note 1, so $(I^i)^k a = I^{ik} a = 0$ by above note 2, and thus $a \in H^0_I(A)$.

Therefore, the double inclusion is shown, and $H^0_I(A) = H^0_J(A)$, as we wished to show.

Exercise 4.6.2 (Mayer-Vietoris sequence) Let I and J be ideals in a noetherian ring R. Show that there is a long exact sequence for every R-module A:

$$\cdots \xrightarrow{\delta} H^q_{I+J}(A) \to H^q_I(A) \oplus H^q_J(A) \to H^q_{I\cap J}(A) \to H^{q+1}_{I+J}(A) \xrightarrow{\delta} \cdots.$$

Hint: Apply $\mathrm{Ext}^*(-, A)$ to the family of sequences

$$0 \to R/I^i \cap J^i \to R/I^i \oplus R/J^i \to R/I^i + J^i \to 0.$$

Then pass to the limit, observing that $(I+J)^{2i} \subseteq (I^i + J^i) \subseteq (I+J)^i$ and that, by the Artin-Rees lemma ([BA II, 7.13]), for every i there is an $N \geq i$ so that $I^N \cap J^N \subseteq (I \cap J)^i \subseteq I^i \cap J^i$.

We follow the hint, which pretty much tells us everything. Applying contravariant $\mathrm{Ext}^*(-, A)$ yields a long exact sequence

$$\cdots \xrightarrow{\delta} \mathrm{Ext}^q\left(R/(I^i + J^i), A\right) \to \mathrm{Ext}^q\left(R/I^i \oplus R/J^i, A\right) \to \mathrm{Ext}^q\left(R/(I^i \cap J^i), A\right) \xrightarrow{\delta} \mathrm{Ext}^{q+1}\left(R/(I^i + J^i), A\right) \to \cdots.$$

Passing to the limit, we have

$$\cdots \xrightarrow{\delta} \varinjlim \mathrm{Ext}^q\left(R/(I^i + J^i), A\right) \to \varinjlim \mathrm{Ext}^q\left(R/I^i \oplus R/J^i, A\right) \to \varinjlim \mathrm{Ext}^q\left(R/(I^i \cap J^i), A\right) \xrightarrow{\delta} \varinjlim \mathrm{Ext}^{q+1}\left(R/(I^i + J^i), A\right) \to \cdots.$$

By Exercise 4.6.1, since $((I+J)^i)^2 \subseteq (I^i + J^i) \subseteq (I+J)^i$, we see that

$$\varinjlim \mathrm{Ext}^q\left(R/(I^i + J^i), A\right) \cong \varinjlim \mathrm{Ext}^q\left(R/(I+J)^i, A\right) = H^q_{I+J}(A).$$

Similarly, since $I^N \cap J^N \subseteq (I \cap J)^i \subseteq I^i \cap J^i$,

$$\varinjlim \operatorname{Ext}^q \left({}^{R}\!\diagup_{(I^i \cap J^i)}, A \right) \cong \varinjlim \operatorname{Ext}^q \left({}^{R}\!\diagup_{(I \cap J)^i}, A \right) = H^q_{I \cap J}(A).$$

Finally, since $\operatorname{Ext}(-, A)$ commutes with a coproduct and limits commute with limits, we have

$$\varinjlim \operatorname{Ext}^q \left({}^{R}\!\diagup_{I^i} \oplus {}^{R}\!\diagup_{J^i}, A \right) \cong \varinjlim \left[\operatorname{Ext}^q \left({}^{R}\!\diagup_{I^i}, A \right) \oplus \operatorname{Ext}^q \left({}^{R}\!\diagup_{J^i}, A \right) \right] \cong \varinjlim \operatorname{Ext}^q \left({}^{R}\!\diagup_{I^i}, A \right) \oplus \varinjlim \operatorname{Ext}^q \left({}^{R}\!\diagup_{J^i}, A \right) = H^q_I(A) \oplus H^q_J(A).$$

Therefore, we have the Mayer-Vietoris long exact sequence

$$\cdots \xrightarrow{\delta} H^q_{I+J}(A) \to H^q_I(A) \oplus H^q_J(A) \to H^q_{I \cap J}(A) \xrightarrow{\delta} H^{q+1}_{I+J}(A) \to \cdots,$$

as desired.

Generalization 4.6.2 (Cohomology with supports; See [GLC]) Let Z be a closed subspace of a topological space X. If F is a sheaf on X, let $H^0_Z(X, F)$ be the kernel of $H^0(X, F) \to H^0(X \setminus Z, F)$, that is, all global sections of F with support in Z. H^0_Z is a left exact functor on Sheaves(X), and we write $H^n_Z(X, F)$ for its right derived functors.

If I is any ideal of R, then $H^n_I(A)$ is defined to be $H^n_Z(X, \widetilde{A})$, where $X = \operatorname{Spec}(R)$ is the topological space of prime ideals of R, $Z = \{\mathfrak{p} \mid I \subseteq \mathfrak{p}\}$, and $\widetilde{A}$ is the sheaf on $\operatorname{Spec}(R)$ associated to A. If I is a finitely generated ideal, this agrees with our earlier definition. For more details see [GLC], including the construction of the long exact sequence

$$0 \to H^0_Z(X, F) \to H^0(X, F) \to H^0(X \setminus Z, F) \to H^1_Z(X, F) \to \cdots.$$

A standard result in algebraic geometry states that $H^n(\operatorname{Spec}(R), \widetilde{A}) = 0$ for $n \neq 0$, so for the *punctured spectrum* $U = \operatorname{Spec}(R) \setminus Z$ the sequence

$$0 \to H^0_I(A) \to A \to H^0(U, \widetilde{A}) \to H^1_I(A) \to 0$$

is exact, and for $n \neq 0$ we can calculate the cohomology of $\widetilde{A}$ on U via

$$H^n(U, \widetilde{A}) \cong H^{n+1}_I(A).$$

Exercise 4.6.3 Let $\mathcal{A}$ be the full subcategory of R-**mod** consisting of the modules with $H^0_I(A) = A$.

1. Show that $\mathcal{A}$ is an abelian category, that $H^0_I : R\text{-}\mathbf{mod} \to \mathcal{A}$ is right adjoint to the inclusion $\iota : \mathcal{A} \hookrightarrow R$-**mod**, and that ι is an exact functor.
2. Conclude that H^0_I preserves injectives (2.3.10), and that $\mathcal{A}$ has enough injectives.
3. Conclude that each $H^n_I(A)$ belongs to the subcategory $\mathcal{A}$ of R-**mod**.

1. Recall that an abelian category is an additive category such that every map has a kernel and cokernel, every monic is the kernel of its cokernel, and every epi is the cokernel of its kernel. An additive category is an **Ab**-category such that it has 0 and finite products. An **Ab**-category is a category such that if we have a diagram

$$A \xrightarrow{f} B \underset{g'}{\overset{g}{\rightrightarrows}} C \xrightarrow{h} D$$

then $h(g+g')f = hgf + hg'f$. So we proceed in reverse, showing $\mathcal{A}$ is **Ab**, additive, and abelian. First, $\mathcal{A}$ is a full subcategory, which means for any objects $A, B \in \mathrm{obj}(\mathcal{A})$, $\mathrm{Hom}_{\mathcal{A}}(A,B) = \mathrm{Hom}_R(A,B)$; i.e., for all objects in the subcategory, we retain all arrows between them. This immediately guarantees that $\mathcal{A}$ is an **Ab**-category, since R-**mod** is. Next, $\mathcal{A}$ is an additive category: it has 0, since $H_I^0(0) = \{a \in 0 \mid I^k a = 0\} = 0$, and it has finite products, since if $A, B \in \mathcal{A}$, i.e., $H_I^0(A) = A$ and $H_I^0(B) = B$, then $H_I^0(A \times B) = A \times B$, since for arbitrary $(a,b) \in A \times B$, we know $I^k a = 0$ for some k and $I^\ell b = 0$ for some ℓ, so since $I^{n+1} \subseteq I^n$ and thus generally, if $r \leq s$, then $I^s \subseteq I^r$, we must have that $I^{\max\{k,\ell\}}(a,b) = 0$ and hence $H_I^0(A \times B) = A \times B$, as desired. Finally, $\mathcal{A}$ is abelian, since again, this is a condition on maps, and $\mathcal{A}$ is a full subcategory, meaning it has any requisite maps from R-**mod**.

Next, we must show that $H_I^0 : R\text{-}\mathbf{mod} \to \mathcal{A}$ is right adjoint to the inclusion functor $\iota : \mathcal{A} \hookrightarrow R\text{-}\mathbf{mod}$. Recall this means we must show

$$\mathrm{Hom}_R(\iota(A), B) \cong \mathrm{Hom}_{\mathcal{A}}(A, H_I^0(B))$$

naturally for all $A \in \mathcal{A}$ and $B \in R$-**mod**. This is immediate though, since $\iota(A) = A = H_I^0(A)$, so given a map $f \in \mathrm{Hom}_R(\iota(A), B) = \mathrm{Hom}_R(A,B)$, we get a map in $\mathrm{Hom}_{\mathcal{A}}(A, H_I^0(B))$ by composing with $H_I^0 : B \to H_I^0(B)$. This has inverse $g \in \mathrm{Hom}_{\mathcal{A}}(A, H_I^0(B))$ maps to ig where $i : H_I^0(B) \hookrightarrow B$ is the natural inclusion. These mappings are indeed inverses, since

$$\mathrm{Hom}_{\mathcal{A}}(A, H_I^0(B)) \longrightarrow \mathrm{Hom}_R(\iota(A), B) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(A, H_I^0(B))$$

$$A \xrightarrow{g} H_I^0(B) \longmapsto A \xrightarrow{g} H_I^0(B) \xrightarrow{i} B \longmapsto A \xrightarrow{g} H_I^0(B) \xrightarrow{i} B \to H_I^0(B) = A \xrightarrow{g} B$$

and

$$\mathrm{Hom}_R(\iota(A), B) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(A, H_I^0(B)) \longrightarrow \mathrm{Hom}_R(\iota(A), B)$$

$$A \xrightarrow{f} B \longmapsto A \xrightarrow{f} B \to H_I^0(B) \longmapsto A \xrightarrow{f} B \to H_I^0(B) \xrightarrow{i} B = A \xrightarrow{f} B$$

Naturality follows by the following commutative diagram, where we are given $A \to A'$ and $B \to B'$:

$$\begin{array}{ccccc} \mathrm{Hom}_R(\iota(A'),B) & \longrightarrow & \mathrm{Hom}_R(\iota(A),B) & \longrightarrow & \mathrm{Hom}_R(\iota(A),B') \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{A}}(A',H_I^0(B)) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(A,H_I^0(B)) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(A,H_I^0(B')) \end{array}$$

The first square commutes because the maps induced by $A \to A'$ and $\iota(A) \to \iota(A')$ are identical, so it makes no difference to do the isomorphism and then the induced map or the induced map and then the isomorphism. The second square commutes because the isomorphism composes with the H_I^0 functor of B, which is a functor and thus respects composition of the induced map $B \to B'$.

Finally, $\iota : \mathcal{A} \to R$-**mod** is an exact functor almost trivially. Given a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, since $\mathcal{A}$ is a full subcategory of R-**mod**, $0 \to \iota(A) \to \iota(B) \to \iota(C) \to 0$ is exactly the same $0 \to A \to B \to C \to 0$, hence a short exact sequence.

2. Proposition 2.3.10 says that if we have a right adjoint functor to an exact functor, then the right adjoint functor preserves injectives; i.e., $R(I)$ is injective whenever I is. Hence by part 1, since H_I^0 is right adjoint to ι which is exact, H_I^0 preserves injectives. We can conclude $\mathcal{A}$ has enough injectives because R-**mod** does and $\mathcal{A}$ is full. Explicitly, let $A \in \mathrm{obj}(\mathcal{A})$, and see that for $\iota(A)$, there exists an injection $0 \to \iota(A) \to J$ with J injective, since R-**mod** has enough injectives. Since H_I^0 is right adjoint, it is left exact, and thus we have $0 \to A \to H_I^0(J)$, and $H_I^0(J)$ is injective. Finally, $H_I^0(J) \in \mathcal{A}$ trivially, since I-torsion of I-torsion is I-torsion, so $H_I^0(H_I^0(J)) = H_I^0(J)$. Therefore $\mathcal{A}$ has enough injectives.

3. We must show that $H_I^n(A) \in \mathcal{A}$ for all n. For any R-module A, to compute $H_I^n(A)$, we take an injective resolution of A, call it $0 \to A \to J^\bullet$, and then take cohomology of $H_I^0(J^\bullet)$. That is,

$$H_I^n(A) = {\ker\left(H_I^0(J^n) \to H_I^0(J^{n+1})\right)} \Big/ {\mathrm{im}\left(H_I^0(J^{n-1}) \to H_I^0(J^n)\right)}.$$

By part 2, $H_I^0(J^\bullet)$ is in $\mathcal{A}$. Since $\mathcal{A}$ is full, $H_I^0(J^n) \to H_I^0(J^{n+1})$ is in $\mathcal{A}$. Since $\mathcal{A}$ is an abelian category by part 1, kernels and cokernels of maps in $\mathcal{A}$ are in $\mathcal{A}$. Thus $\ker\left(H_I^0(J^n) \to H_I^0(J^{n+1})\right)$ is in $\mathcal{A}$, and $H_I^n(A)$, a quotient of the kernel, i.e., a cokernel, must be in $\mathcal{A}$, as desired.

Theorem 4.6.3 *Let R be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$. Then the grade $G(A)$ of any finitely generated R-module A is the smallest integer n such that $H_{\mathfrak{m}}^n(A) \neq 0$.*

Proof. For each i we have the exact sequence

$$\operatorname{Ext}^{n-1}\left(\mathfrak{m}^i/\mathfrak{m}^{i+1}, A\right) \to \operatorname{Ext}^n\left(R/\mathfrak{m}^i, A\right) \to \operatorname{Ext}^n\left(R/\mathfrak{m}^{i+1}, A\right) \to \operatorname{Ext}^n\left(\mathfrak{m}^i/\mathfrak{m}^{i+1}, A\right).$$

We saw in 4.4.8 that $\operatorname{Ext}^n\left(R/\mathfrak{m}, A\right)$ is zero if $n < G(A)$ and nonzero if $n = G(A)$; as $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is a finite direct sum of copies of $R/\mathfrak{m}$, the same is true for $\operatorname{Ext}^n\left(\mathfrak{m}^i/\mathfrak{m}^{i+1}, A\right)$. By induction on i, this proves that $\operatorname{Ext}^n\left(R/\mathfrak{m}^{i+1}, A\right)$ is zero if $n < G(A)$ and that it contains the nonzero module $\operatorname{Ext}^n\left(R/\mathfrak{m}^i, A\right)$ if $n = G(A)$. Now take the direct limit as $i \to \infty$. $\square$

Application 4.6.4 Let R be a 2-dimensional local domain. Since $G(R) \neq 0$, $H^0_{\mathfrak{m}}(R) = 0$. From the exact sequence

$$0 \to \mathfrak{m}^i \to R \to R/\mathfrak{m}^i \to 0$$

we obtain the exact sequence

$$0 \to R \to \operatorname{Hom}_R(\mathfrak{m}^i, R) \to \operatorname{Ext}^1_R\left(R/\mathfrak{m}^i, R\right) \to 0.$$

As R is a domain, there is a natural inclusion of $\operatorname{Hom}_R(\mathfrak{m}^i, R)$ in the field F of fractions of R as the submodule

$$\mathfrak{m}^{-1} \equiv \{x \in F \mid x\mathfrak{m}^i \subseteq R\}.$$

Set $C = \cup \mathfrak{m}^{-i}$. (*Exercise*: Show that C is a subring of F.) Evidently

$$H^1_{\mathfrak{m}}(R) = \varinjlim \operatorname{Ext}^1\left(R/\mathfrak{m}^i, R\right) \cong C/R.$$

If R is Cohen-Macaulay, that is, $G(R) = 2$, then $H^1_{\mathfrak{m}}(R) = 0$, so $R = C$ and $\operatorname{Hom}_R(\mathfrak{m}^i, R) = R$ for all i. Otherwise $R \neq C$ and $G(R) = 1$. When the integral closure of R is finitely generated as an R-module, C is actually a Cohen-Macaulay ring - the smallest Cohen-Macaulay ring containing R [EGA, IV.5.10.17].

Here is an alternative construction of local cohomology due to Serre [EGA, III.1.1]. If $x \in R$ there is a natural map from $K(x^{i+1})$ to $K(x^i)$:

$$\begin{array}{ccccccc}
0 & \longrightarrow & R & \xrightarrow{x^{i+1}} & R & \longrightarrow & 0 \\
 & & {\scriptstyle x}\downarrow & & \| & & \\
0 & \longrightarrow & R & \xrightarrow{x^i} & R & \longrightarrow & 0.
\end{array}$$

By tensoring these maps together, and writing $\boldsymbol{x}^i$ for $({x_1}^i, \cdots, {x_n}^i)$, this gives a map from $K(\boldsymbol{x}^{i+1})$ to $K(\boldsymbol{x}^i)$, hence a tower $\{H_q(K(\boldsymbol{x}^i))\}$ of R-modules. Applying $\operatorname{Hom}_R(-, A)$ and taking cohomology yields a map from $H^q(\boldsymbol{x}^i, A)$ to $H^q(\boldsymbol{x}^{i+1}, A)$.

Definition 4.6.5 $H^q_{\boldsymbol{x}}(A) = \varinjlim H^q(\boldsymbol{x}^i, A)$.

For our next result, recall from 3.5.6 that a tower $\{A_i\}$ satisfies the *trivial Mittag-Leffler condition* if for every i there is a $j > i$ so that $A_j \to A_i$ is zero.

Exercise 4.6.4 If $\{A_i\} \to \{B_i\} \to \{C_i\}$ is an exact sequence of towers of R-modules and both $\{A_i\}$ and $\{C_i\}$ satisfy the trivial Mittag-Leffler condition, then $\{B_i\}$ also satisfies the trivial Mittag-Leffler condition (3.5.6).

Recall a tower $\{M_i\}$ is of the form $\cdots \to M_2 \to M_1 \to M_0$. Let k be arbitrary; we seek $j > k$

such that $B_j \to B_k$ is zero. Since $\{A_i\}$ and $\{C_i\}$ satisfy the trivial Mittag-Leffler condition, there exist j_A and j_C greater than k such that $A_{j_A} \to A_k$ is zero and $C_{j_C} \to C_k$ is zero. Thus let $j = \max\{j_A, j_C\}$ and we have $A_j \to A_k$ is $A_j \to A_{j_A} \to A_k$ the zero map and similarly $C_j \to C_k$ is $C_j \to C_{j_C} \to C_k$ the zero map. Thus since $\{A_i\} \to \{B_i\} \to \{C_i\}$ is an exact sequence, we have the commutative diagram

$$\begin{tikzcd}
A_j \arrow[r] \arrow[d, "0"'] & B_j \arrow[r] \arrow[d] & C_j \arrow[d, "0"] \\
A_k \arrow[r] & B_k \arrow[r] & C_k
\end{tikzcd}$$

Since the first square commutes, $\begin{tikzcd}[cramped] A_j \arrow[r] & B_j \arrow[d] \\ & B_k \end{tikzcd}$ is the zero map. If $B_j \to B_k$ is the zero map, we are done.

Suppose to the contrary it is not. By the commutativity of the second square, $\begin{tikzcd}[cramped] B_j \arrow[d] & \\ B_k \arrow[r] & C_k \end{tikzcd}$ is the zero map, and since $B_j \to B_k$ is nonzero, $B_k \to C_k$ must take elements in the image of $B_j \to B_k$ to 0. By exactness of $\{A_i\} \to \{B_i\} \to \{C_i\}$, this forces the image of $A_j \to A_k$ to surject onto the image of $B_j \to B_k$. Yet the image of $A_j \to A_k$ is 0, so $B_j \to B_k$ must be the zero map, contradicting our supposition it was not.

Since k was arbitrary, $\{B_i\}$ satisfies the trivial Mittag-Leffler condition, as desired.

Proposition 4.6.6 *Let R be a commutative noetherian ring and A a finitely generated R-module. Then the tower $\{H_q(\boldsymbol{x}^i, A)\}$ satisfies the trivial Mittag-Leffler condition for every $q \neq 0$.*

Proof. We proceed by induction on the length n of $\boldsymbol{x}$. If $n = 1$, one sees immediately that $H_1(x^i, A)$ is the submodule $A_i = \{a \in A \mid x^i a = 0\}$. The submodules A_i of A form an ascending chain, which must be stationary since R is noetherian and A is finitely generated. This means that there is an integer k such that $A_k = A_{k+1} = \cdots$, that is, $x^k A_i = 0$ for all i. Since the map $A_{i+j} \to A_i$ is multiplication by x^j, it is zero whenever $j \geq k$. Thus the lemma holds if $n = 1$.

Inductively, set $\boldsymbol{y} = (x_1, ..., x_{n-1})$ and write x for x_n. Since $K(\boldsymbol{x}^i) \otimes K(\boldsymbol{y}^i) = K(\boldsymbol{x}^i)$, the Künneth formula for Koszul complexes 4.5.3 (and its proof) yields the following exact sequence of towers:

$$\{H_q(\boldsymbol{y}^i, A)\} \to \{H_q(\boldsymbol{x}^i, A)\} \to \{H_{q-1}(\boldsymbol{y}^i, A)\};$$
$$\{H_1(\boldsymbol{y}^i, A)\} \to \{H_1(\boldsymbol{x}^1, A)\} \to \left\{H_1\left(x^i, {}^{A}\!/\!_{\boldsymbol{y}^i A}\right)\right\} \to 0.$$

If $q \geq 2$, the outside towers satisfy the trivial Mittag-Leffler condition by induction, so $\{H_q(\boldsymbol{x}^i, A)\}$ does too. If $q = 1$ and we set $A_{ij} = \left\{a \in {}^{A}\!/\!_{\boldsymbol{y}^i A} \mid x^j a = 0\right\} = H_1\left(x^j, {}^{A}\!/\!_{\boldsymbol{y}^i A}\right)$, it is enough to show that the diagonal tower $\{A_{ii}\}$ satisfies the trivial Mittag-Leffler condition. For fixed i, we saw above that there is a k such that every map $A_{ij} \to A_{i,j+k}$ is zero. Hence the map $A_{ii} \to A_{i,i+k} \to A_{i+k,i+k}$ is zero, as desired. $\square$

Corollary 4.6.7 *Let R be commutative noetherian, and let E be an injective R-module. Then $H^q_{\boldsymbol{x}}(E) = 0$ for all $q \neq 0$.*

Proof. Because E is injective, $\mathrm{Hom}_R(-, E)$ is exact. Therefore

$$H^q(\boldsymbol{x}^i, E) = H^q \,\mathrm{Hom}_R(K(\boldsymbol{x}^i, R), E) \cong \mathrm{Hom}_R(H_q(\boldsymbol{x}^i, R), R).$$

Because the tower $\{H_q(\boldsymbol{x}^i, R)\}$ satisfies the trivial Mittag-Leffler condition,

$$H^q_{\boldsymbol{x}}(E) \cong \varinjlim \mathrm{Hom}_R(H_q(\boldsymbol{x}^i, R), E) = 0.$$

$\square$

Theorem 4.6.8 *If R is commutative noetherian, $\boldsymbol{x} = (x_1, \cdots, x_n)$ is any sequence of elements of R, and $I = (x_1, \cdots, x_n)R$, then for every R-module A*

$$H^q_I(A) \cong H^q_{\boldsymbol{x}}(A).$$

Proof. Both H^q_I and $H^q_{\boldsymbol{x}}$ are universal δ-functors, and

$$H^0_I(A) = \varinjlim \mathrm{Hom}\left({R}/{\boldsymbol{x}^i R}, A\right) = \varinjlim H^0(\boldsymbol{x}^i, A) = H^0_{\boldsymbol{x}}(A).$$

$\square$

Corollary 4.6.9 *If R is a noetherian local ring, then $H^q_{\mathfrak{m}}(A) \neq 0$ only when $G(A) \leq q \leq \dim(R)$. In particular, if R is a Cohen-Macaulay local ring, then*

$$H^q_{\mathfrak{m}}(R) \neq 0 \iff q = \dim(R).$$

Proof. Set $d = \dim(R)$. By standard commutative ring theory ([KapCR, Thm.153]), there is a sequence $\boldsymbol{x} = (x_1, \cdots, x_d)$ of elements of $\mathfrak{m}$ such that $\mathfrak{m}^j \subseteq I \subseteq \mathfrak{m}$ for some j, where $I = (x_1, \cdots, x_d)R$. But then $H^d_{\mathfrak{m}}(A) = H^q_I(A) = H^q_{\boldsymbol{x}}(A)$, and this vanishes for $q > d$ because the Koszul complexes $K(\boldsymbol{x}^i)$ have length d. Now use (4.6.3). $\square$

Exercise 4.6.5 If I is a finitely generated ideal of R and $R \to S$ is a ring map, show that $H^q_I(A) \cong H^q_{IS}(A)$ for every S-module A. This result is rather surprising, because there isn't any nice relationship between the groups $\mathrm{Ext}^*_R\left({R}/{I^i}, A\right)$ and $\mathrm{Ext}^*_S\left({S}/{I^i}, A\right)$. Consequently, if $\mathrm{ann}_R(A)$ denotes $\{r \in R \mid rA = 0\}$, then $H^q_I(A) = 0$ for $q > \dim\left({R}/{\mathrm{ann}_R(A)}\right)$.

Let $I = (x_1, ..., x_n)R$ for generators $x_1, ..., x_n$, and let $y_1, ..., y_n$ be the images of $x_1, ..., x_n$ in S, so that the ideal $IS = (y_1, ..., y_n)S$. By Theorem 4.6.8, $H^q_I(A) \cong H^q_{\boldsymbol{x}}(A)$ and $H^q_{IS}(A) \cong H^q_{\boldsymbol{y}}(A)$. Since by definition,

$$H^q_{\boldsymbol{x}}(A) = \varinjlim H^q(\boldsymbol{x}^i, A) = \varinjlim H^q(\mathrm{Hom}_R(K(\boldsymbol{x}^i), A)) \text{ and}$$
$$H^q_{\boldsymbol{y}}(A) = \varinjlim H^q(\boldsymbol{y}^i, A) = \varinjlim H^q(\mathrm{Hom}_S(K(\boldsymbol{y}^i), A)),$$

and we may think of an S-module A as an R-module by restriction of scalars by $R \to S$, it is enough to show that we may identify Koszul cohomologies. By Exercise 4.5.2, we may work with Koszul homology $H_p(\boldsymbol{x}, A) = H_p(K(\boldsymbol{x}) \otimes_R A)$ instead, since we have duality isomorphisms.

Thus, see that

$$K(\boldsymbol{x}) \otimes_R A \cong (K(\boldsymbol{x}) \otimes_R S) \otimes_S A \cong K(\boldsymbol{y}) \otimes_S A,$$

so the homologies agree, as desired, and thus the result is shown.

Application 4.6.10 (Hartshorne) Let $R = \mathbf{C}[x_1, x_2, y_1, y_2]$, $P = (x_1, x_2)R$, $Q = (y_1, y_2)R$, and $I = P \cap Q$. As P, Q, and $\mathfrak{m} = P + Q = (x_1, x_2, y_1, y_2)R$ are generated by regular sequences, the outside terms in the Mayer-Vietoris sequence (exercise 4.6.2)

$$H^3_P(R) \oplus H^3_Q(R) \to H^3_I(R) \to H^4_{\mathfrak{m}}(R) \to H^4_P(R) \oplus H^4_Q(R)$$

vanish, yielding $H^3_I(R) \cong H^4_{\mathfrak{m}}(R) \neq 0$. This implies that the union of two planes in $\mathbf{C}^4$ that meet in a point cannot be described as the solution of only two equations $f_1 = f_2 = 0$. Indeed, if this were the case, then we would have $I^i \subseteq (f_1, f_2)R \subseteq I$ for some i, so that $H^3_I(R)$ would equal $H^3_f(R)$ which is zero.